

Geometric Theorems and Postulates

This version is similar to the previously posted and still available in the website listing a cross reference with Geometric Standards for California. Both the ones before the Common core and the Common Core.

It is a work in progress and might have some errors that will be corrected and we spot them.

Negation:

The negation is the negation of a statement

$\sim p$ is "No p " or negating " p "

An Angle is Right

p



An Angle is Not Right

$\sim p$

An Angle is Not Right

p



An Angle is Right

$\sim p$

Counter – Example:

Determine the validity of the CONJECTURE. If it is false give a COUNTER EXAMPLE

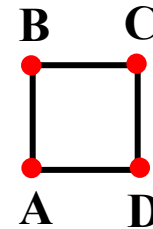
Given:

Points A, B, C, D

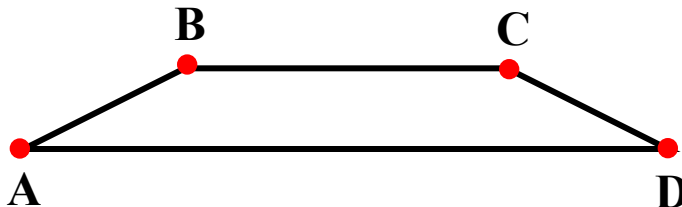
Conjecture:

It is only possible to arrange points in a square

False!



Counter – Example:



It is possible to arrange them in a trapezoid

Conditional Statements or Propositions:

if ... then ...

$$\textcolor{red}{p} \rightarrow \textcolor{blue}{q}$$

if { $\textcolor{red}{p}$ } then { $\textcolor{blue}{q}$ }

Where:

p = hypothesis

q = conclusion

Converse:

if ... then ...

$$\textcolor{blue}{q} \rightarrow \textcolor{red}{p}$$

if { $\textcolor{blue}{q}$ } then { $\textcolor{red}{p}$ }

Where:

p = conclusion

q = hypothesis

Inverse:

if ... then ...

$$\sim \textcolor{red}{p} \rightarrow \sim \textcolor{blue}{q}$$

if { $\sim \textcolor{red}{p}$ } then { $\sim \textcolor{blue}{q}$ }

Where:

p = hypothesis

q = conclusion

Contrapositive:

if ... then ...

$$\sim \textcolor{blue}{q} \rightarrow \sim \textcolor{red}{p}$$

if { $\sim \textcolor{blue}{q}$ } then { $\sim \textcolor{red}{p}$ }

Where:

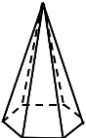
p = conclusipn

q = hypothesis

~ = negation

Law of Detachment:

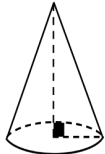
If $\{p \rightarrow q\}$ is a true statement and p is true then $\{q \text{ is true.}\}$
General $\rightarrow$ **Particular**



$$\text{If } \left\{ S.A. = \frac{1}{2} \cdot P \cdot l + \frac{1}{2} \cdot P \cdot a \right\}$$

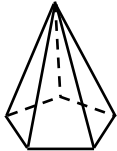
$$\text{then } \left\{ S.A. = \frac{1}{2} \cdot 36 \cdot 26.5 + \frac{1}{2} \cdot 36 \cdot 3 \cdot \sqrt{3} \rightarrow S.A. = 15 \cdot \pi \approx \mathbf{810} \right\}$$

with $P = 36$ and $l = 26.5$ and $a = 3 \cdot \sqrt{3}$



$$\text{If } \{S.A. = \pi \cdot r^2 + \pi \cdot r \cdot l\}$$

$$\text{then } \{S.A. = \pi \cdot (3)^2 + \pi \cdot 3 \cdot 2 \rightarrow S.A. = \mathbf{15 \cdot \pi}\} \text{ with } r = 3 \text{ and } l = 2$$



$$\text{If } \left\{ V = \frac{1}{3} \cdot B \cdot h \right\} \text{ then } \left\{ V = \frac{1}{3} \cdot 2 \cdot 3 \rightarrow V = \mathbf{2} \right\} \text{ with } B = 2 \text{ and } h = 3$$



$$\text{If } \{S.A. = 4 \cdot \pi \cdot r^2\} \text{ then } \{S.A. = 4 \cdot \pi \cdot (3)^2 \rightarrow S.A. = \mathbf{36 \cdot \pi}\} \text{ with } r = 3$$

$$\text{If } \left\{ V = \frac{4}{3} \cdot \pi \cdot r^3 \right\} \text{ then } \left\{ V = \frac{4}{3} \cdot \pi \cdot (3)^3 \rightarrow S.A. = \mathbf{12 \cdot \pi} \right\} \text{ with } r = 3$$

Law of Sylogism:

If { $p \rightarrow q$ } and { $q \rightarrow r$ } then { $p \rightarrow r$ }

***If {  $\rightarrow$ 2 pairs of $\parallel$ Sides } and { A parallelogram $\rightarrow$ 2 pairs of $\parallel$ Sides }
then {  $\rightarrow$ A parallelogram }***

***If { $\angle ABC \rightarrow$ measures 60° } and { $0^\circ < \text{angle} < 90^\circ \rightarrow$ An acute $\angle$ }
then { $\angle ABC \rightarrow$ Acute $\angle$ }***

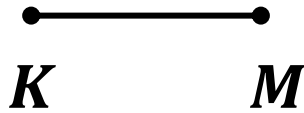
***If { A kite $\rightarrow$ 2 pairs of consecutive sides $\cong' s$ }
and { ABCD $\rightarrow$ 2 pairs of consecutive sides $\cong' s$ }
then { ABCD $\rightarrow$ KITE }***

Recall:

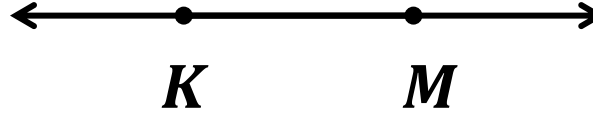
Transitive Porperty in Algebra: if { $a = b$ } and { $b = c$ } then { $a = c$ }

Segments, lines and rays

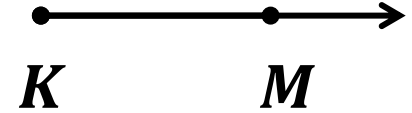
Segment $\overline{KM}$



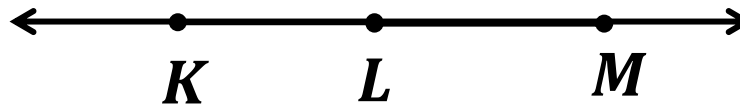
Line $\overleftrightarrow{KM}$



Ray $\overrightarrow{KM}$



Segment Addition Postulate:



if {Point L is between points K and M in $\overline{KM}$ }

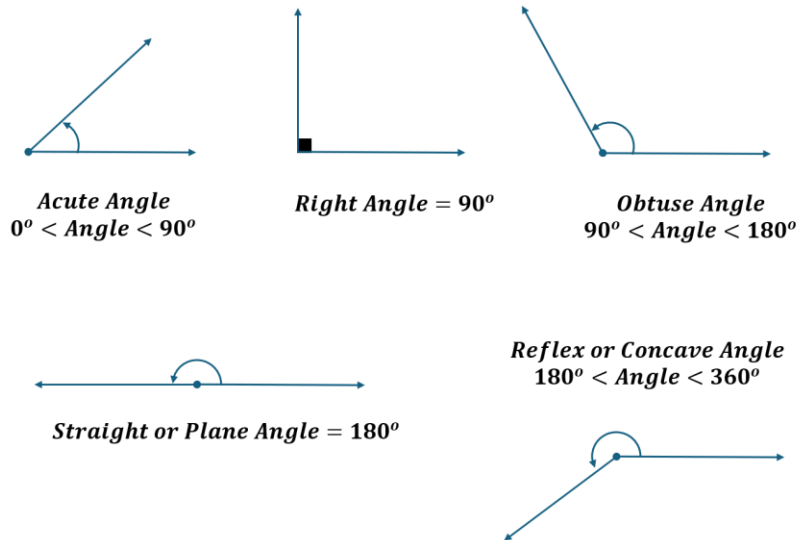
then $\{\overline{KM} \cong \overline{KL} + \overline{LM} \rightarrow KM = KL + LM\}$

Or

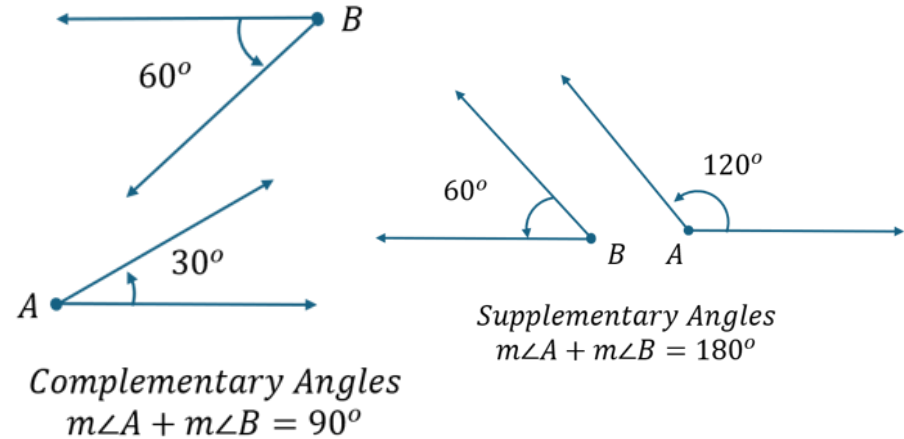
if $\{\overline{KM} \cong \overline{KL} + \overline{LM} \rightarrow KM = KL + LM\}$

then {Point L is between points K and M in $\overline{KM}$ }

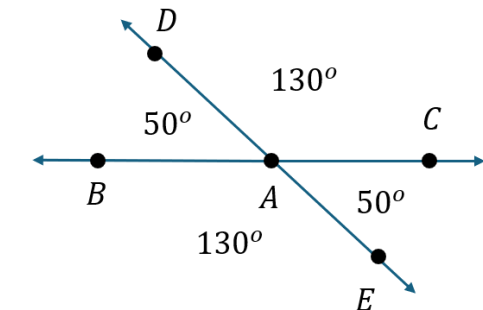
Angle Classification



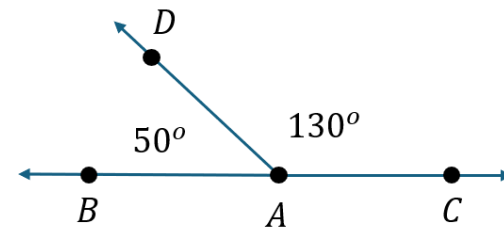
Angle Pair Relationships



Angle Pair Relationships

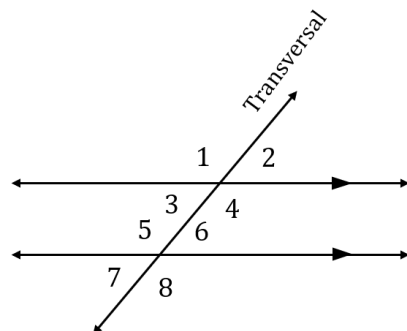


Vertical Angles:
Opposite by the Vertex
Congruent Angles



Linear Pair:
Supplementary Angles
Adjacent Angles
Common Ray $\overrightarrow{AD}$

Angle Pair Relationships in $\parallel$ Lines Cut by a Transversal



Corresponding Angles

$$\angle 1 \cong \angle 5$$

$$\angle 2 \cong \angle 6$$

$$\angle 3 \cong \angle 7$$

$$\angle 4 \cong \angle 8$$

Alternate Interior Angles

$$\angle 3 \cong \angle 6$$

$$\angle 4 \cong \angle 5$$

Alternate Exterior Angles

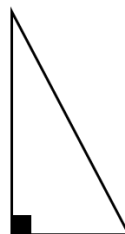
$$\angle 1 \cong \angle 8$$

$$\angle 2 \cong \angle 7$$

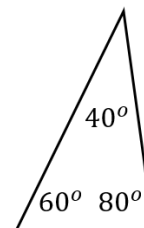
Same Side Interior Angles or Consecutive Interior Angles

$$m\angle 3 + m\angle 5 = 180^\circ \quad \& \quad m\angle 4 + m\angle 6 = 180^\circ$$

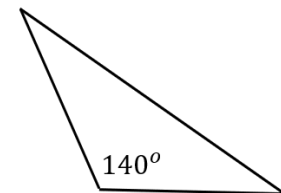
Triangle Clasifcation by Angles



Right Triangle:
One Right Angle

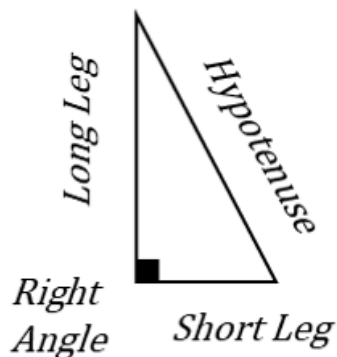


Acute Triangle:
All Angles are Acute

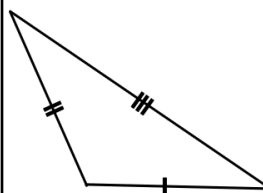


Obtuse Triangle:
One Angle is Obtuse

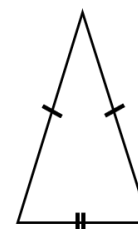
Partes de un Triángulo Recto



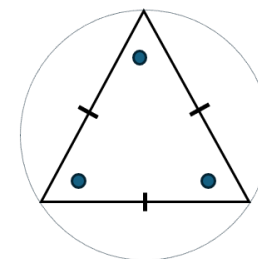
Triangle Clasifcation by Sides



Scalene Triangle:
All Sides Different

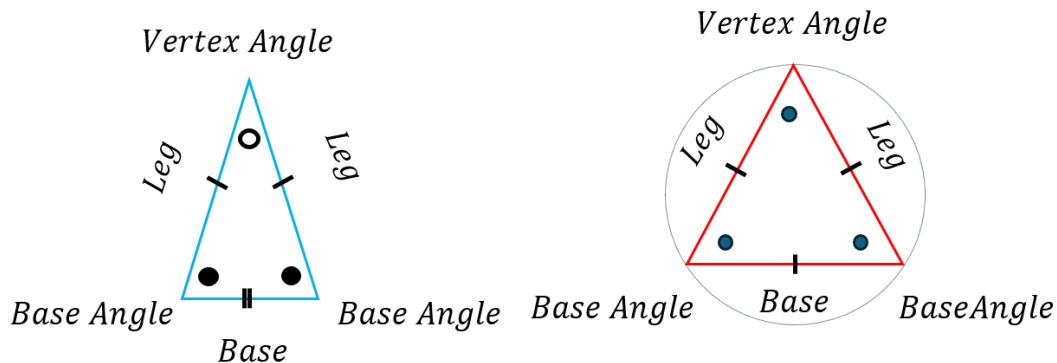


Isosceles Triangle:
At Least Two
Congruent Sides



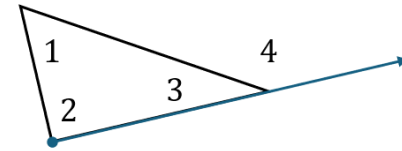
Equilateral Triangle:
Three Congruent Sides
● *Also Equiangular*

Parts of a Isosceles Triangle



All **equilateral triangles** are **isosceles triangles** but not all isosceles triangles are equilateral.

Exterior Angle Theorem



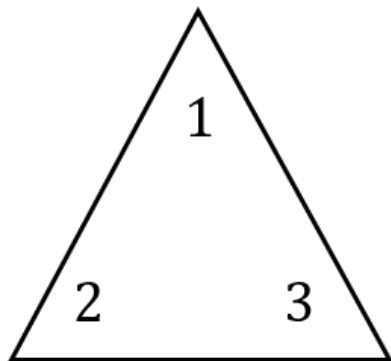
$\angle 1$ & $\angle 2 \rightarrow$ Remote Interior $\angle$'s

$\angle 3 \rightarrow$ Interior No Adjacent $\angle$ to Exterior $\angle$

$\angle 4 \rightarrow$ Exterior $\angle$

Exterior Angle is Equal to the Sum of Its Two Remote Interior Angles
 $m\angle 4 = m\angle 1 + m\angle 2$

Interior Angle Sum in Triangles



$$m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$$

Proof of the Sum of Interior Angles in Regular and Irregular Concave Polygons

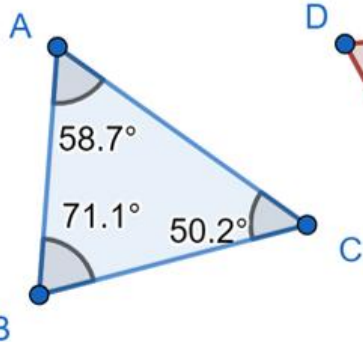
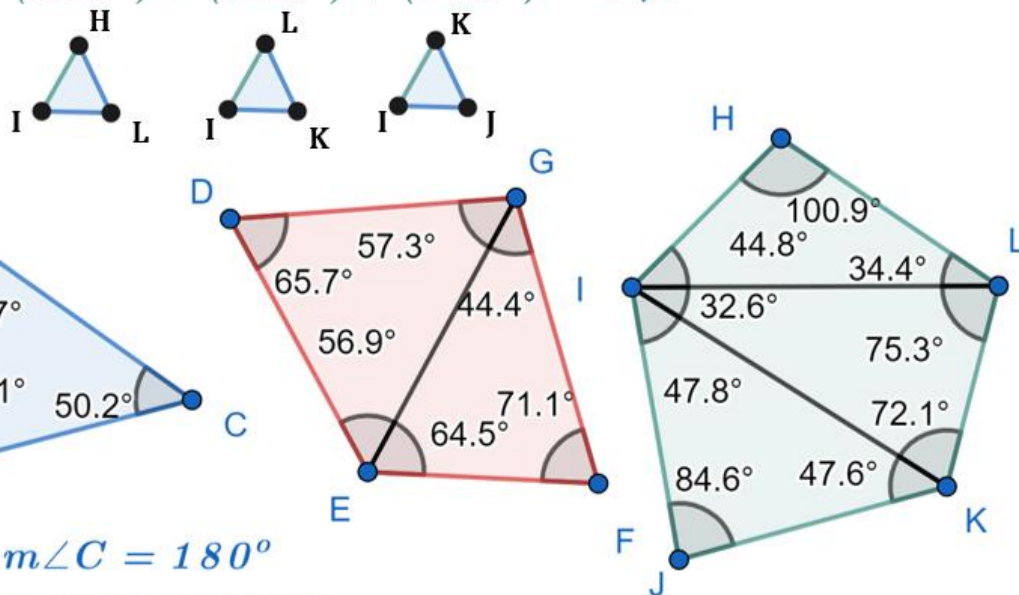
$$m\angle H + m\angle I + m\angle J + m\angle K + m\angle L = 540^\circ$$

$$m\angle H + (m\angle LIH + m\angle KIL + m\angle JIK) + m\angle J + (m\angle LKI + m\angle IKJ) + (m\angle HLI + m\angle ILK) = 540^\circ$$

$$100.9^\circ + (44.8^\circ + 32.6^\circ + 47.8^\circ) + 84.6^\circ + (47.6^\circ + 72.1^\circ) + (75.3^\circ + 34.4^\circ) = 540^\circ$$

$$(100.9^\circ + 44.8^\circ + 34.4^\circ) + (32.6^\circ + 75.3^\circ + 72.1^\circ) + (47.6^\circ + 84.6^\circ + 47.8^\circ) = 540^\circ$$

$$(180^\circ) + (180^\circ) + (180^\circ) = 540^\circ$$



$$m\angle A + m\angle B + m\angle C = 180^\circ$$

$$58.7^\circ + 71.1^\circ + 50.2^\circ = 180^\circ$$

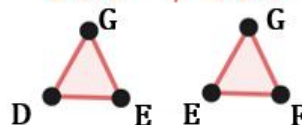
$$m\angle D + m\angle E + m\angle F + m\angle G = 360^\circ$$

$$m\angle D + (m\angle GED + m\angle FEG) + m\angle F + (m\angle DGE + m\angle EGF) = 360^\circ$$

$$65.7^\circ + (56.9^\circ + 57.3^\circ) + 71.1^\circ + (44.4^\circ + 64.5^\circ) = 360^\circ$$

$$(65.7^\circ + 56.9^\circ + 57.3^\circ) + (71.1^\circ + 44.4^\circ + 64.5^\circ) = 360^\circ$$

$$180^\circ + 180^\circ = 360^\circ$$

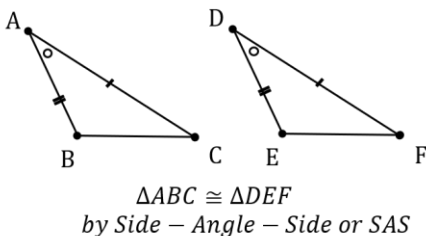
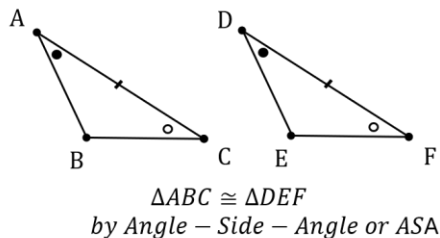
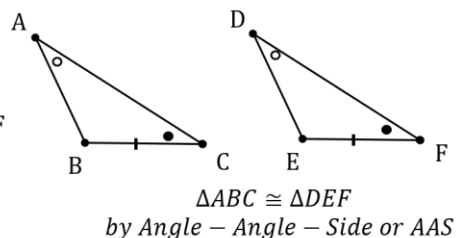
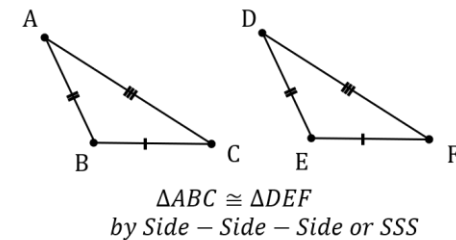


S = Interior Angle Sum in Polygons
= (Number of Triangles) $\cdot 180^\circ$

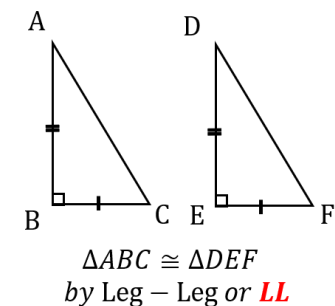
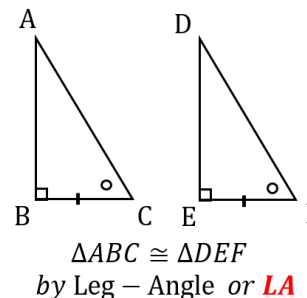
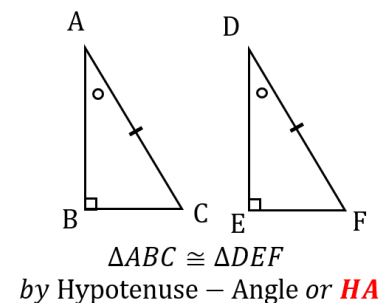
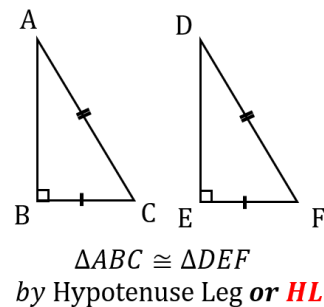
Number Triangles = Number Sides $- 2$
 n = Number Sides

$$S = (n - 2) \cdot 180^\circ$$

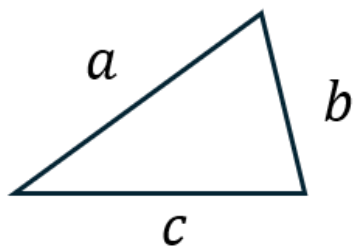
Triangle Congruence Theorems and Postulates



Triangle Classification by Angles



Triangle Inequality

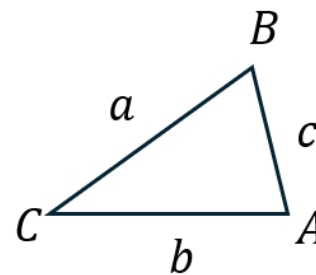


$$a < b + c$$

$$b < a + c$$

$$c < a + b$$

Triangle Inequality: Ordering Angles



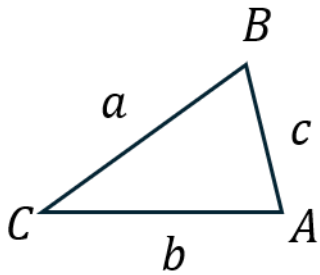
$$\text{if } \{a > b > c\}$$

then

$$\{m\angle A > m\angle B > m\angle C\}$$

Triangle Inequality: Ordering Sides

if $\{m\angle A > m\angle B > m\angle C\}$

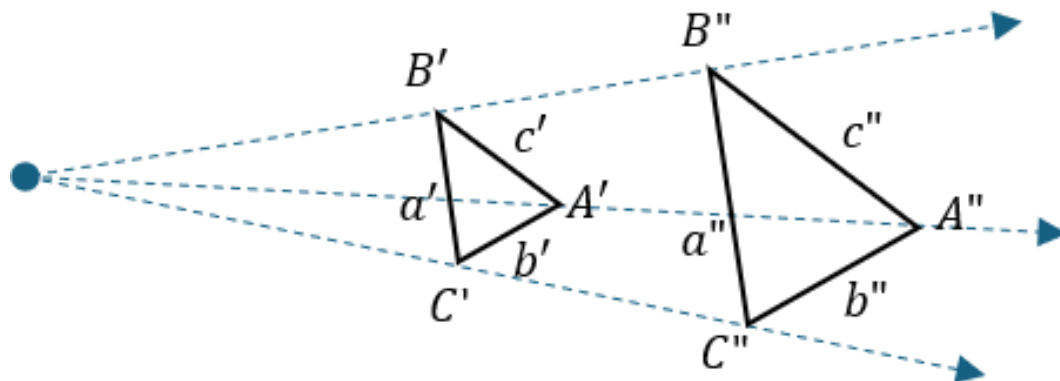


then

$\{a > b > c\}$

Similarity in Triangles

Corresponding Sides are
Proportional
or
in Proportion



$$\text{If } \left\{ \begin{array}{l} \frac{a'}{a''} = \frac{b'}{b''} = \frac{c'}{c''} \\ \text{or} \\ \frac{a''}{a'} = \frac{b''}{b'} = \frac{c''}{c'} \end{array} \right\} \& \left\{ \begin{array}{l} \angle A' \cong \angle A'' \\ \angle B' \cong \angle B'' \\ \angle C' \cong \angle C'' \end{array} \right\}$$

Then

$$\{\triangle A'B'C' \sim \triangle A''B''C''\}$$

$\sim \rightarrow$ **Similarity Symbol**

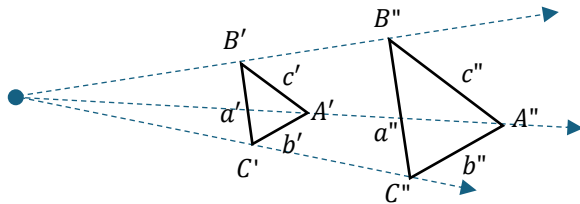
Corresponding Angles
Congruent

The Converse is also true

Proportional Reasoning and Similarity in Triangles

Side–Side–Side or SSS ~

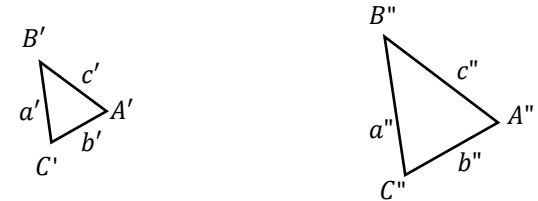
$$\text{If } \left\{ \begin{array}{l} \frac{a'}{a''} = \frac{b'}{b''} = \frac{c'}{c''} \\ \text{or} \\ \frac{a''}{a'} = \frac{b''}{b'} = \frac{c''}{c'} \end{array} \right\} \quad \text{Then } \left\{ \begin{array}{l} \Delta A'B'C' \sim \Delta A''B''C'' \\ \text{by SSS } \sim \end{array} \right\}$$



The converse is true as well

Angle – Angle ~

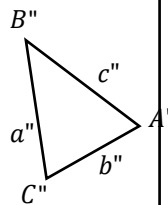
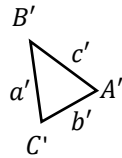
$$\text{If } \left\{ \begin{array}{l} \angle A' \cong \angle A'' \\ \angle B' \cong \angle B'' \\ \text{or} \\ \angle A' \cong \angle A'' \\ \angle C' \cong \angle C'' \\ \text{or} \\ \angle B' \cong \angle B'' \\ \angle C' \cong \angle C'' \end{array} \right\} \quad \text{Then } \left\{ \begin{array}{l} \Delta A'B'C' \sim \Delta A''B''C'' \\ \text{by AA } \sim \end{array} \right\}$$



The converse is true as well

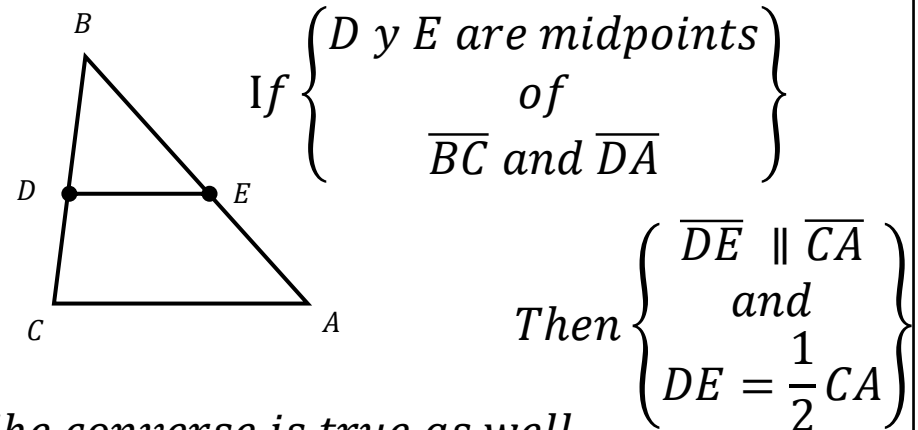
Side–Angle–Side or SAS ~

$$\text{If } \left\{ \begin{array}{l} \frac{a'}{a''} = \frac{c'}{c''} \\ \text{or} \\ \frac{a''}{a'} = \frac{c''}{c'} \\ \text{and} \\ \angle B' \cong \angle B'' \end{array} \right\} \quad \text{Then } \left\{ \begin{array}{l} \Delta A'B'C' \sim \Delta A''B''C'' \\ \text{by SAS } \sim \end{array} \right\}$$



The converse is true as well

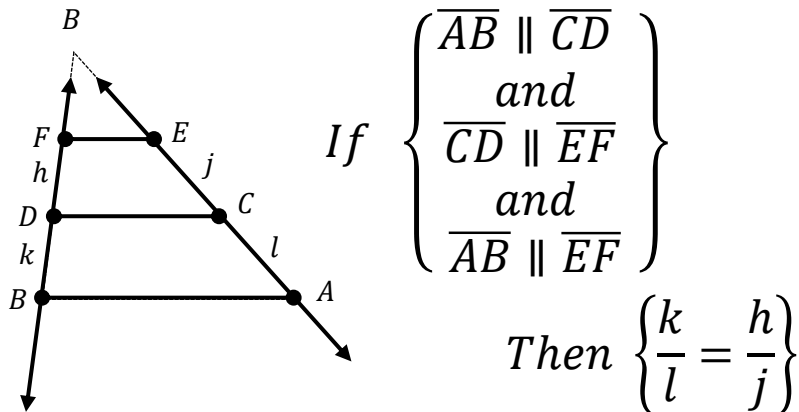
Two Sides Connected at Midpoint in a Δ and the third one $\parallel$ to this.



The converse is true as well

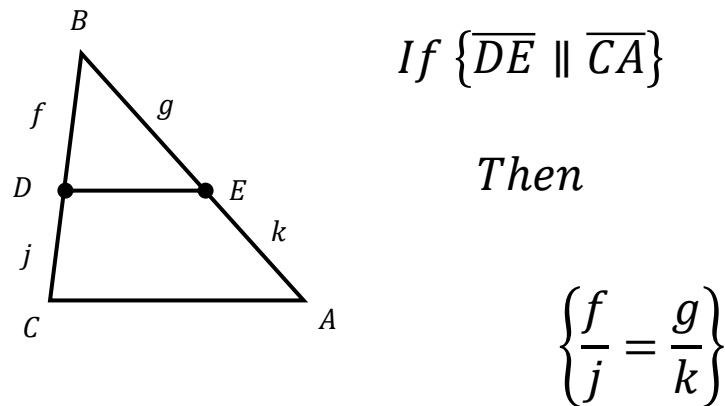
Proportional Reasoning and Similarity in Triangles

Proportionality in several $\parallel$ lines cut by transversals



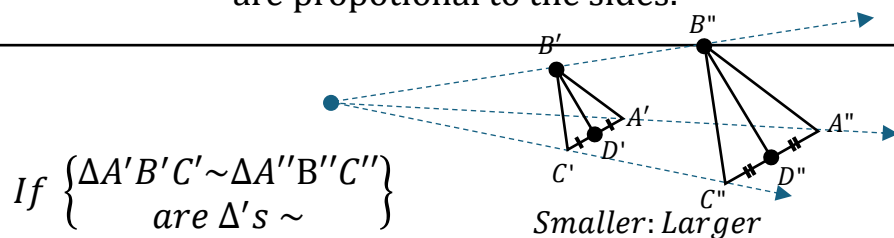
The converse is true as well

Proportionality in two sides in a Δ joined by a segment $\parallel$ to the third side.



The converse is true as well

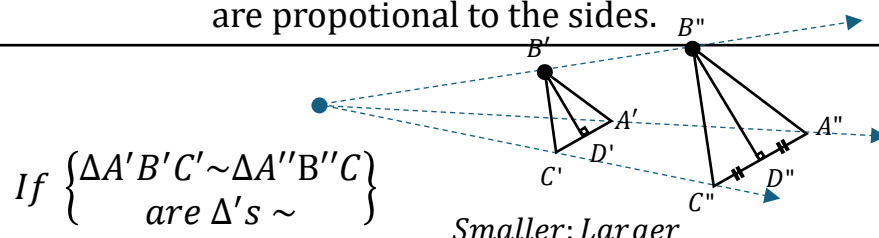
Medians $\overline{B'D'}$ and $\overline{B''D''}$ in similar triangles are proportional to the sides.



Then $\left\{ \begin{array}{l} \frac{A'B'}{A''B''} = \frac{B'C'}{B''C''} = \frac{C'A'}{C''A''} = \frac{B'D'}{B''D''} \\ \text{or} \\ \frac{A''B''}{A'B'} = \frac{B''C''}{B'C'} = \frac{C''A''}{C'A'} = \frac{B''D''}{B'D'} \end{array} \right\}$

Larger: Smaller

Altitudes or Heights $\overline{B'D'}$ and $\overline{B''D''}$ in similar triangles are proportional to the sides.

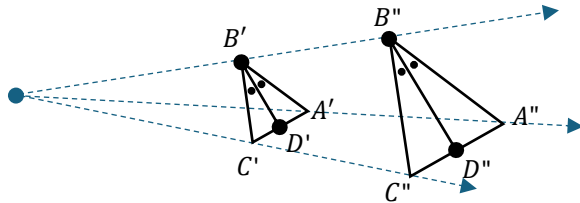


Then $\left\{ \begin{array}{l} \frac{A'B'}{A''B''} = \frac{B'C'}{B''C''} = \frac{C'A'}{C''A''} = \frac{B'D'}{B''D''} \\ \text{or} \\ \frac{A''B''}{A'B'} = \frac{B''C''}{B'C'} = \frac{C''A''}{C'A'} = \frac{B''D''}{B'D'} \end{array} \right\}$

Larger: Smaller

Proportional Reasoning and Similarity in Triangles

Angle Bisectors $\overline{B'D'}$ and $\overline{B''D''}$ in similar triangles are proportional to the sides.



If $\left\{ \begin{array}{l} \Delta A'B'C' \sim \Delta A''B''C'' \\ \text{are } \Delta's \sim \end{array} \right\}$ *Smaller: Larger*

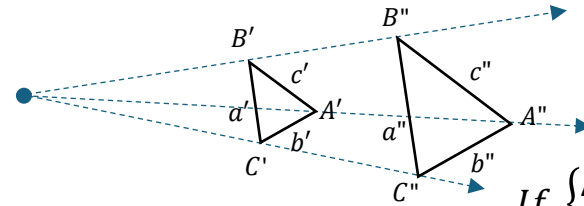
Then $\left\{ \begin{array}{l} \frac{A'B'}{A''B''} = \frac{B'C'}{B''C''} = \frac{C'A'}{C''A''} = \frac{B'D'}{B''D''} \\ \text{or} \\ \frac{A''B''}{A'B'} = \frac{B''C''}{B'C'} = \frac{C''A''}{C'A'} = \frac{B''D''}{B'D'} \end{array} \right\}$

Larger: Smaller

Perimeters

$$a' + b' + c' \quad \text{and} \quad a'' + b'' + c''$$

in similar triangles are proportional to the sides.

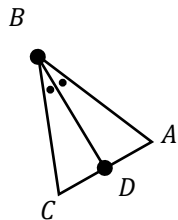


If $\left\{ \begin{array}{l} \Delta A'B'C' \sim \Delta A''B''C'' \\ \text{are } \Delta's \sim \end{array} \right\}$ *Smaller: Larger*

Then $\left\{ \begin{array}{l} \frac{A'B'}{A''B''} = \frac{B'C'}{B''C''} = \frac{C'A'}{C''A''} = \frac{a' + b' + c'}{a'' + b'' + c''} \\ \text{or} \\ \frac{A''B''}{A'B'} = \frac{B''C''}{B'C'} = \frac{C''A''}{C'A'} = \frac{a'' + b'' + c''}{a' + b' + c'} \end{array} \right\}$

Larger: Smaller

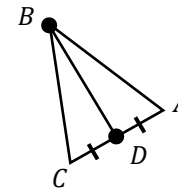
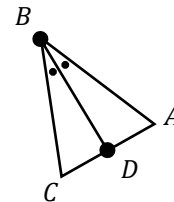
Segments Formed by Angle Bisector from One Vertex in Opposite Side are Proportional to Corresponding Adjacent Sides



$$\left\{ \frac{CD}{DA} = \frac{BC}{BA} \right\}$$

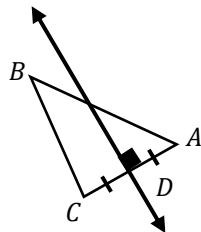
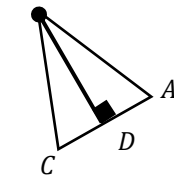
Special Segments, Lines, Rays in Triangles

Angle Bisector



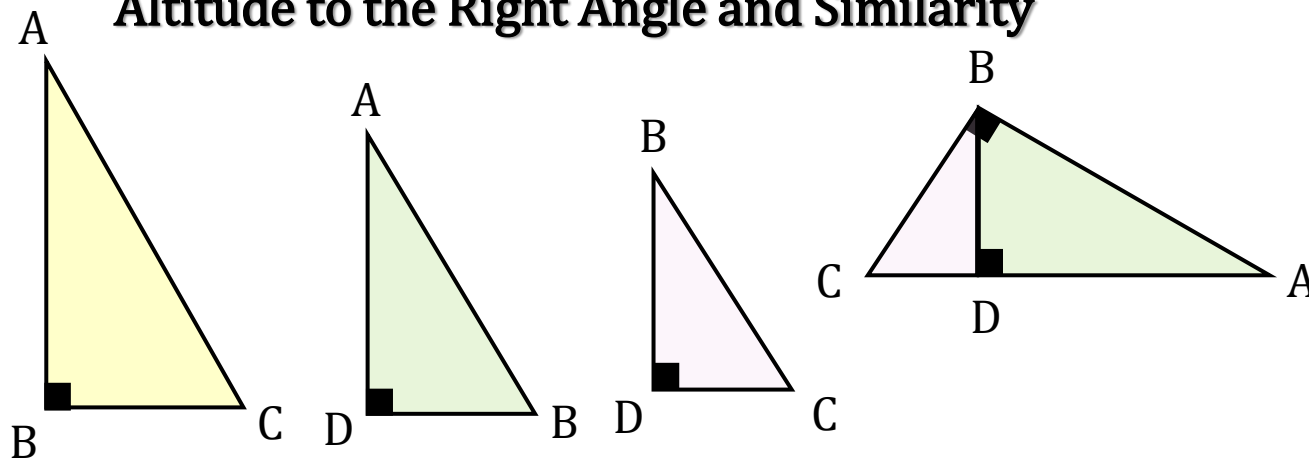
Median

Altitude or Height



Perpendicular Bisector

Altitude to the Right Angle and Similarity



The highlighted proportions involve geometric means; knowing that the product of the means equals the product of the extremes, we derive the three geometric mean theorems associated with the altitude to the right angle.

$$\triangle ABC \sim \triangle ADB$$

$$\frac{AB}{AD} = \frac{AC}{AB}$$

$$\frac{AB}{AD} = \frac{AC}{AB}$$

$$\frac{AB}{AD} = \frac{AC}{AB}$$

$$\frac{AB}{AD} = \frac{AC}{AB}$$

$$\frac{AB}{AD} = \frac{AC}{AB}$$

$$\frac{AB}{AD} = \frac{AC}{AB}$$

$$\frac{AB}{AD} = \frac{AC}{AB}$$

$$\frac{AB}{AD} = \frac{AC}{AB}$$

$$AB^2 = AC \cdot AD$$

The geometric mean of the longer leg is equal to the product of the hypotenuse and the adjacent segment of the hypotenuse formed by the altitude to the right angle.

$$\triangle ADB \sim \triangle BDC$$

$$\frac{AD}{DB} = \frac{AB}{BC}$$

$$\frac{AD}{DB} = \frac{AB}{BC}$$

$$\frac{AD}{DB} = \frac{AB}{BC}$$

$$\frac{AD}{DB} = \frac{AB}{BC}$$

$$\frac{AD}{DB} = \frac{AB}{BC}$$

$$\frac{AD}{DB} = \frac{AB}{BC}$$

$$\frac{AD}{DB} = \frac{AB}{BC}$$

$$\frac{AD}{DB} = \frac{AB}{BC}$$

$$DB^2 = AD \cdot DC$$

The geometric mean of the altitude to the right angle is equal to the product of the segments of the hypotenuse formed by the altitude to the right angle.

$$\triangle ABC \sim \triangle BDC$$

$$\frac{AB}{BD} = \frac{AC}{BC}$$

$$\frac{AB}{BD} = \frac{AC}{BC}$$

$$\frac{AB}{BD} = \frac{AC}{BC}$$

$$\frac{AB}{BD} = \frac{AC}{BC}$$

$$\frac{AB}{BD} = \frac{AC}{BC}$$

$$\frac{AB}{BD} = \frac{AC}{BC}$$

$$\frac{AB}{BD} = \frac{AC}{BC}$$

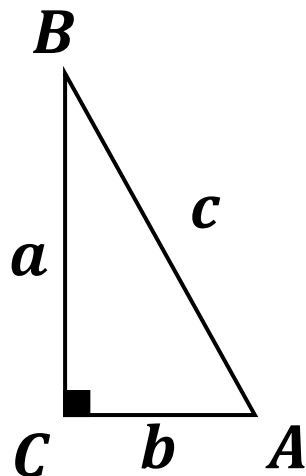
$$\frac{AC}{BC} = \frac{BC}{DC}$$

$$BC^2 = AC \cdot DC$$

The geometric mean of the shorter leg is equal to the product of the hypotenuse and the adjacent segment of the hypotenuse formed by the altitude to the right angle.

Right Triangles: Pythagorean Theorem

Pythagorean theorem



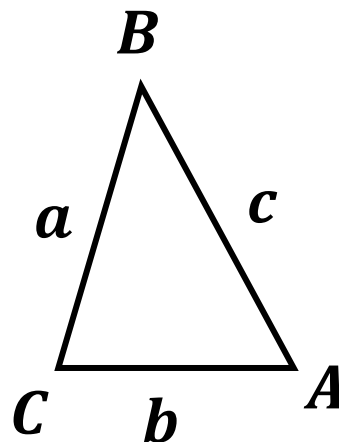
$$c^2 = a^2 + b^2$$

$$|c| = \sqrt{a^2 + b^2}$$

$$c = \pm \sqrt{a^2 + b^2}$$

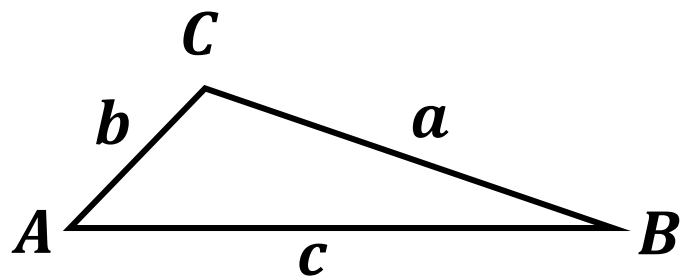
(-) No sense here

Pythagorean theorem *Acute* Δ 's



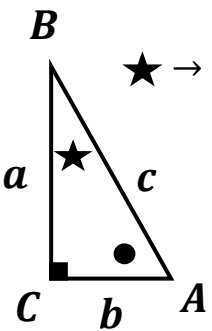
$$c^2 < a^2 + b^2$$

Pythagorean theorem *Obtuse* Δ 's

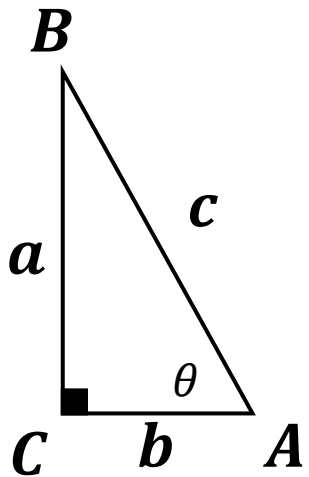


$$c^2 > a^2 + b^2$$

Right Triangles: Trigonometry

Trigonometry Reference Angle	Trigonometry: Right Triangles Bold text used at higher levels	
● → <i>Reference Angle</i> <i>a</i> → <i>Opposite Side</i> <i>b</i> → <i>Adjacent Side</i>  ★ → <i>Reference Angle</i> <i>b</i> → <i>Opposite Side</i> <i>a</i> → <i>Adjacent Side</i>	● → <i>Reference Angle</i> $\sin A = \frac{a}{c} \quad \cot A = \frac{b}{a}$ $\cos A = \frac{b}{c} \quad \sec A = \frac{c}{b}$ $\tan A = \frac{a}{b} \quad \csc A = \frac{c}{a}$	★ → <i>Reference Angle</i> $\sin B = \frac{b}{c} \quad \cot B = \frac{a}{b}$ $\cos B = \frac{a}{c} \quad \sec B = \frac{c}{a}$ $\tan B = \frac{b}{a} \quad \csc B = \frac{c}{b}$

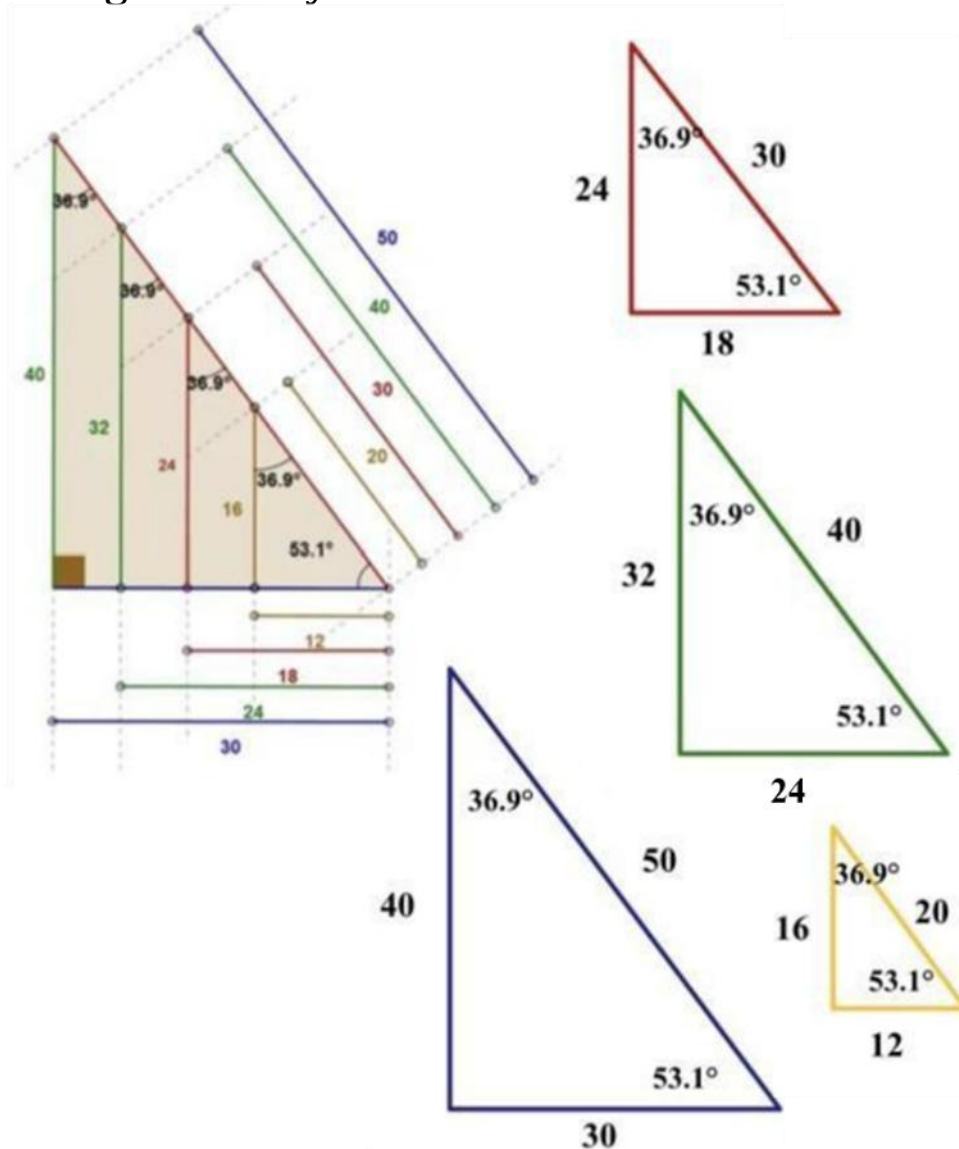
Trigonometría: Triángulos Rectos

	$\alpha, \beta, \gamma, \delta, \theta \rightarrow$ <i>It is common to use them as angles.</i> $\sin \theta = \frac{\text{Opposite Side}}{\text{Hypotenuse}}$ $\cos \theta = \frac{\text{Adjacent Side}}{\text{Hypotenuse}}$ $\tan \theta = \frac{\text{Opposite Side}}{\text{Adjacent Side}}$	$\cot \theta = \frac{\text{Adjacent Side}}{\text{Opposite Side}}$ $\sec \theta = \frac{\text{Hypotenuse}}{\text{Adjacent Side}}$ $\csc \theta = \frac{\text{Hypotenuse}}{\text{Opposite Side}}$
--	--	---

Right Triangles: Trigonometry and Similitary Link

Establishing the similarity of triangles that have congruent corresponding angles and corresponding sides that are proportional, or relating this to trigonometric ratios or trigonometry.

Similarity Proof



$$\sin(53.1^\circ) = \frac{16}{20} = \frac{24}{30} = \frac{32}{40} = \frac{40}{50} = \frac{4}{5} \approx 0.7997$$

$$\cos(53.1^\circ) = \frac{12}{20} = \frac{18}{30} = \frac{24}{40} = \frac{30}{50} = \frac{3}{5} \approx 0.6004$$

$$\tan(53.1^\circ) = \frac{16}{12} = \frac{24}{18} = \frac{32}{24} = \frac{40}{30} = \frac{4}{3} \approx 1.332$$

$$\cot(53.1^\circ) = \frac{12}{16} = \frac{18}{24} = \frac{24}{32} = \frac{30}{40} = \frac{3}{4} \approx 0.7508$$

$$\sec(53.1^\circ) = \frac{20}{12} = \frac{30}{18} = \frac{40}{24} = \frac{50}{30} = \frac{5}{3} \approx 1.666$$

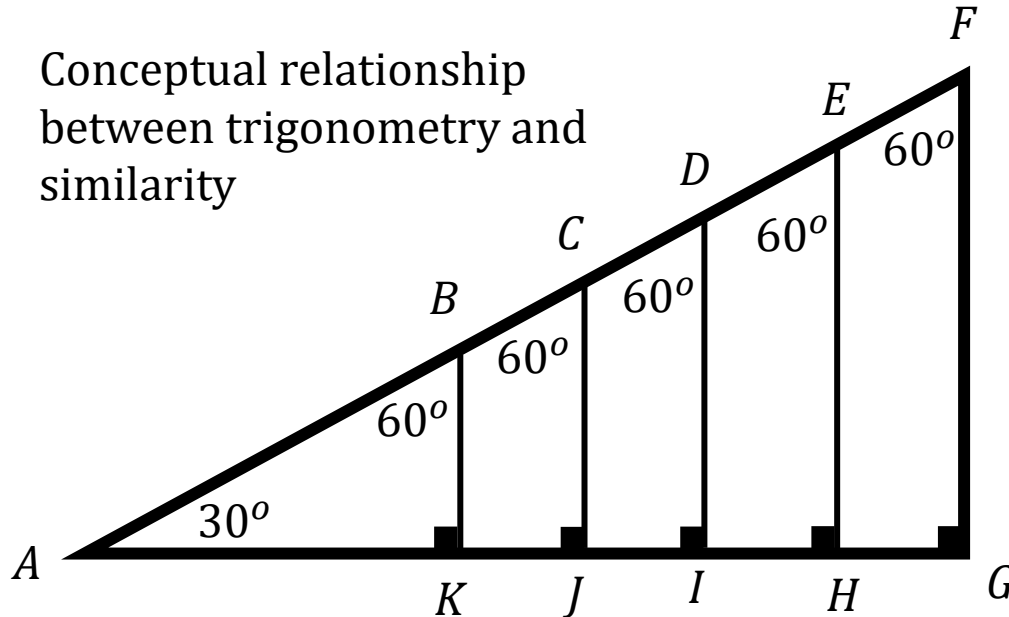
$$\csc(53.1^\circ) = \frac{20}{16} = \frac{30}{24} = \frac{40}{32} = \frac{50}{40} = \frac{5}{4} = 1.25$$

All Δ 's are ($\sim$) by AA $\sim$

Right Triangles: Trigonometry and Similitary Link

Trigonometría de Triángulos Rectos y (~)

Conceptual relationship
between trigonometry and
similarity



$$\triangle ABK \sim \triangle ACJ \sim \triangle ADI \sim \triangle AEH \sim \triangle AFG$$

All embedded triangles are similar.

$$30^\circ - 60^\circ - 90^\circ$$

Scalene right triangles.

They all yield the same numerical
value for each of the six trigonometric
ratios (**angle-to-ratio**) and all produce
the same corresponding angle (**ratio-
to-angle**).

Angle to Trigonometric Ratio

$$\sin(30^\circ) = 0.5$$

$$\cos(30^\circ) \approx 0.87$$

$$\tan(30^\circ) \approx 0.58$$

Trigonometric Ratio to Angle

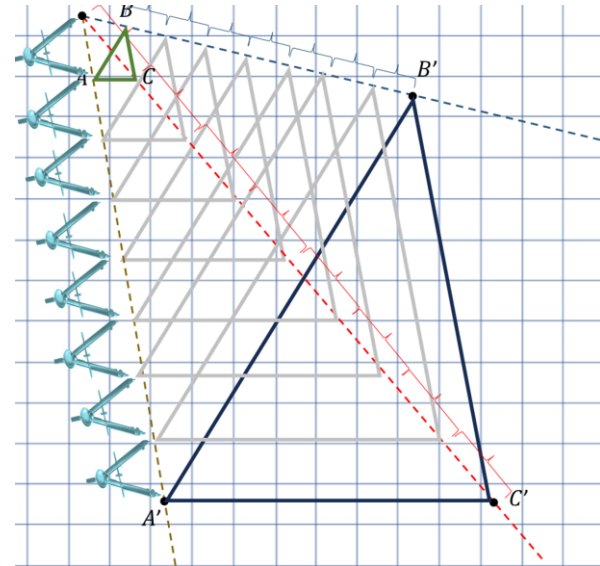
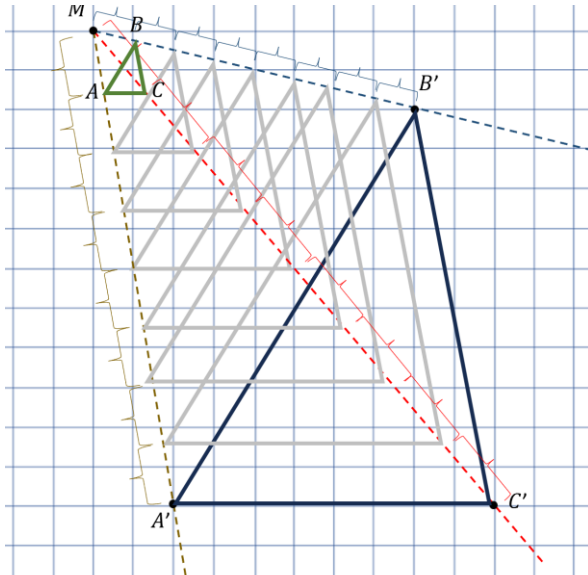
$$\sin^{-1}(0.5) = 30^\circ$$

$$\cos^{-1}(0.87) \approx 30^\circ$$

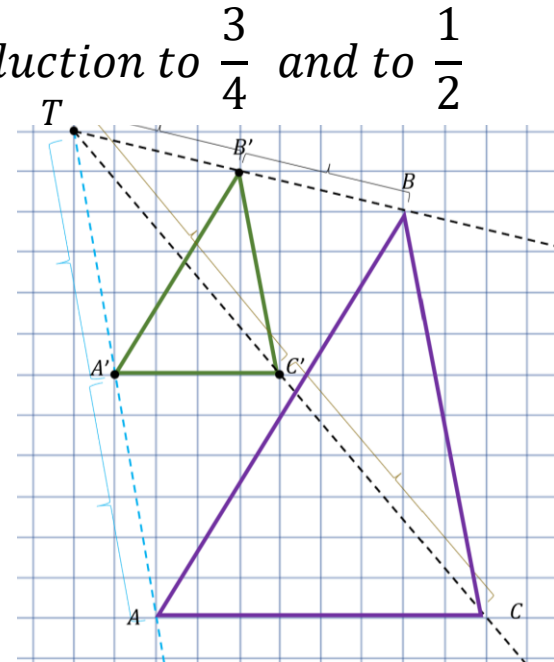
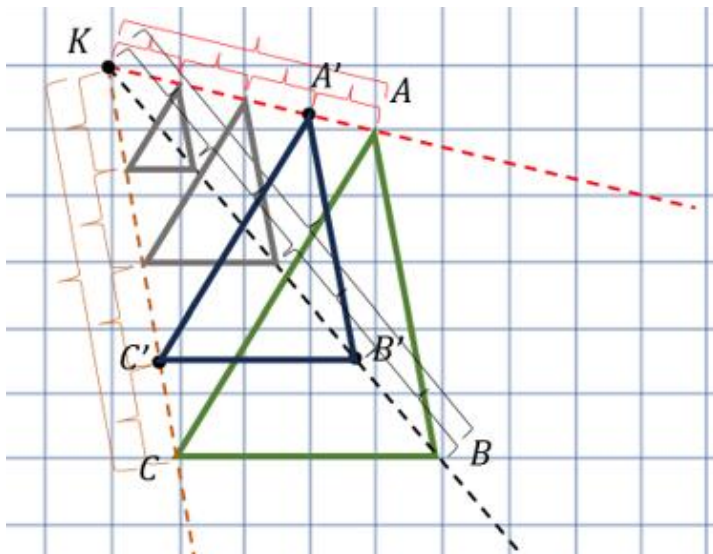
$$\tan^{-1}(0.58) \approx 30^\circ$$

Similarity vs Homothety: Enlargements and Reductions

Performing an elongation by a factor of 8 with center of dilation at point M

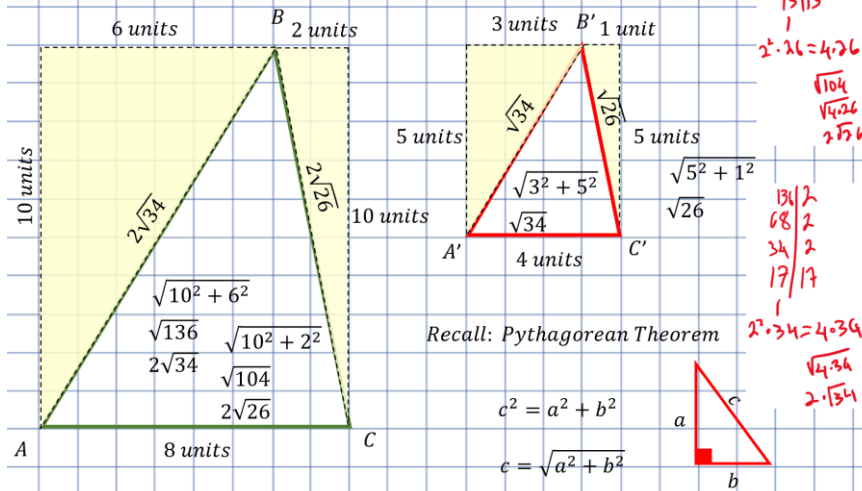


Performing Fractional Dilations: Reduction to $\frac{3}{4}$ and to $\frac{1}{2}$



Similarity: Establishing Extended Proportions and Corresponding Angles $\cong$'s

- Last problem we proved how to get a fractional dilation with a scale of $1:\frac{1}{2}$ with center of dilation T. Now let's prove they are similar with corresponding sides proportional and corresponding angles congruent (Angles are with the same orientation)

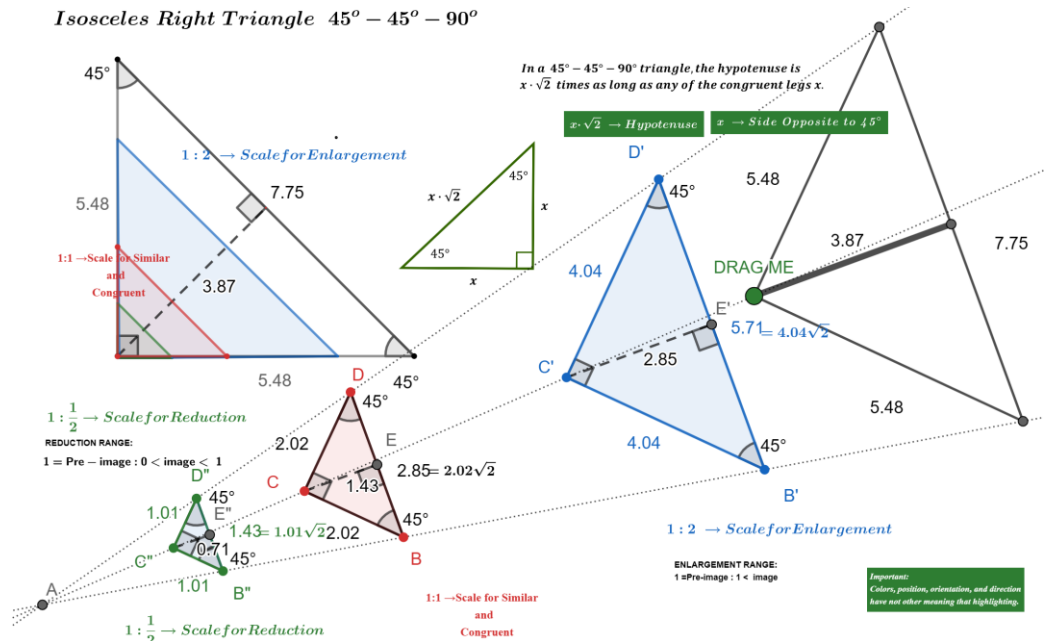


Similarity: Homothety from a center of homothety or dilation for special triangles

Several instances of a special triangle

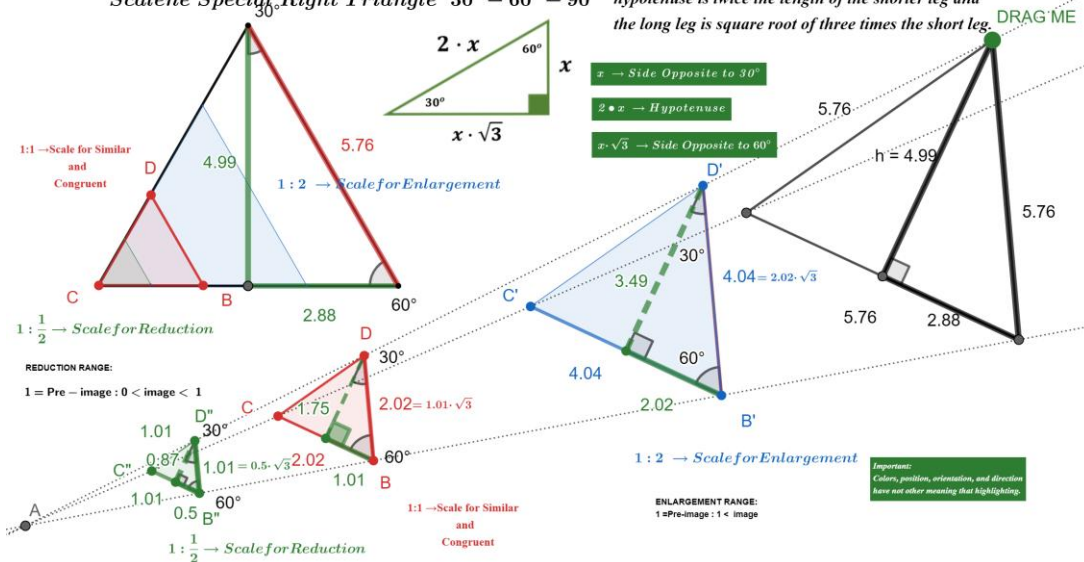
$$45^\circ - 45^\circ - 90^\circ$$

using a center of homothety, distinguishing between enlargement and reduction, and presenting cases of congruence and similarity



Scalene Special Right Triangle $30^\circ - 60^\circ - 90^\circ$

In a scalene special right triangle $30^\circ - 60^\circ - 90^\circ$ the hypotenuse is twice the length of the shorter leg and the long leg is square root of three times the short leg.

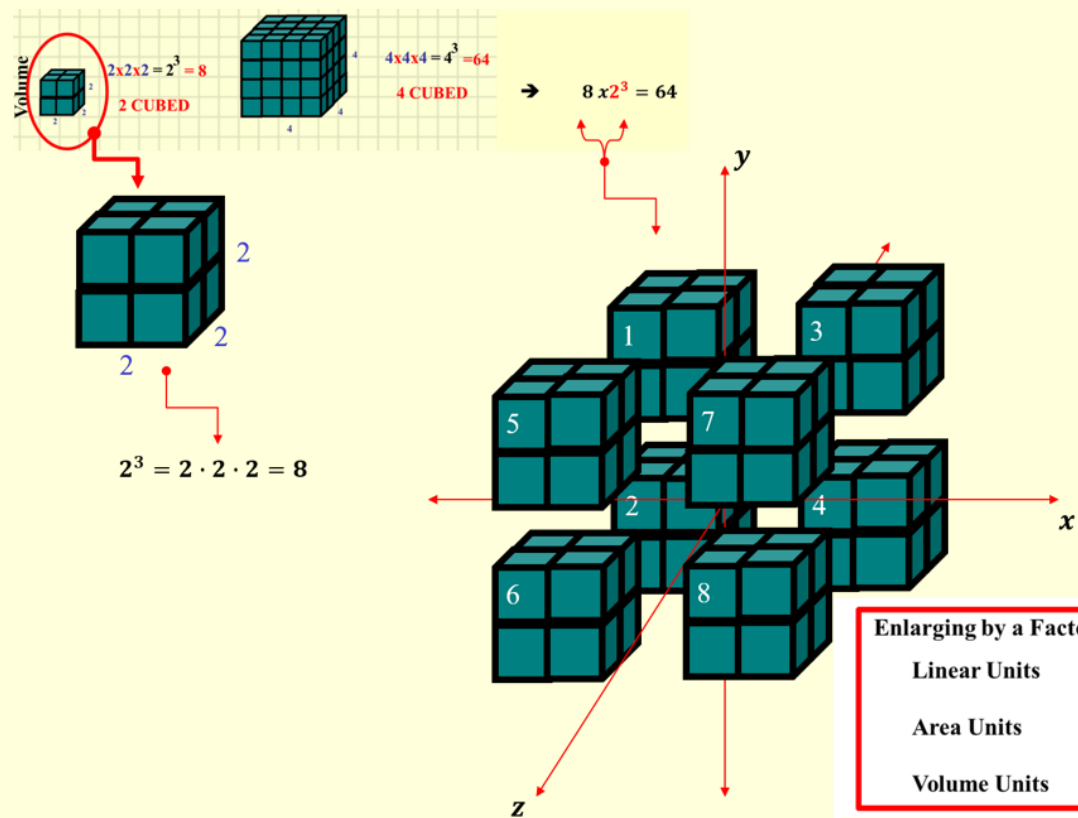


Several instances of a special triangle

$$30^\circ - 60^\circ - 90^\circ$$

using a center of homothety, distinguishing between enlargement and reduction, and presenting cases of congruence and similarity

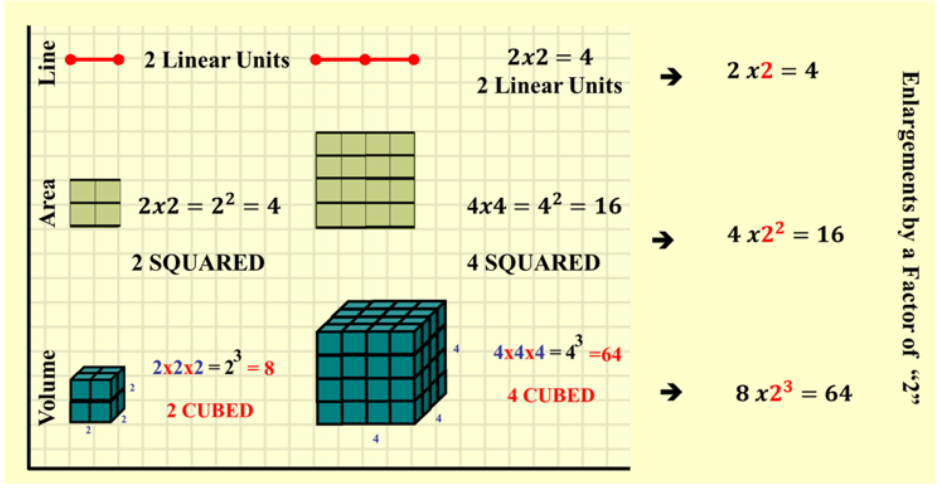
Scale factor in Similarity or using it in non-congruent transformations



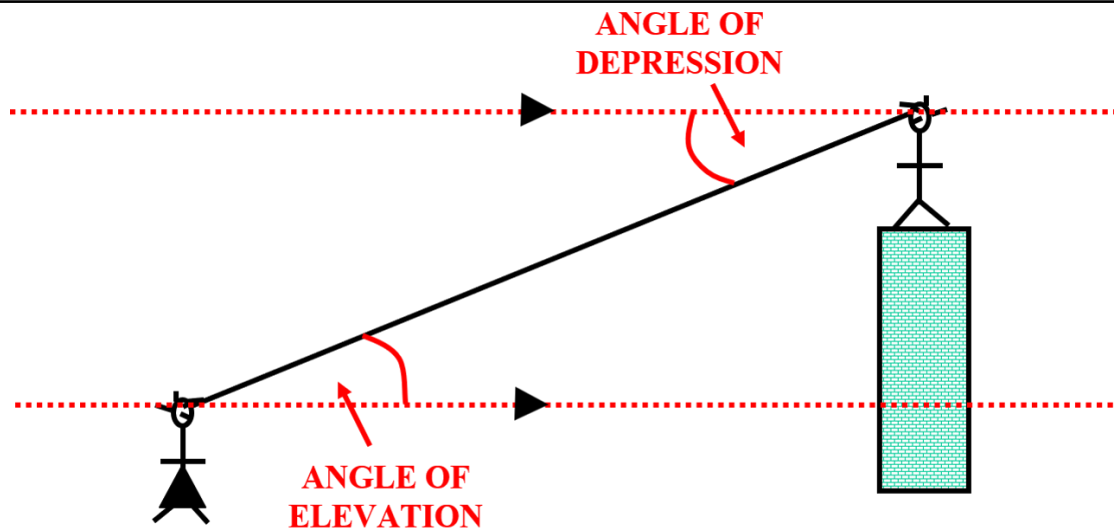
It is necessary to take into account that making dilations of line dimensions is different of doing it of area units or volumen units. Special considerations need to be done if we are going to use as scale factor linear, area or volume units

Enlarging by a Factor of "a"		
Linear Units	→	Multiply by a
Area Units	→	Multiply by a^2
Volume Units	→	Multiply by a^3

Enlargements or reductions of a line, an area, or a volume need to be done as indicated in the left. You multiply by the factor of enlargement to the first power for linear units, to the second power for areas, and to the third power for volumes.

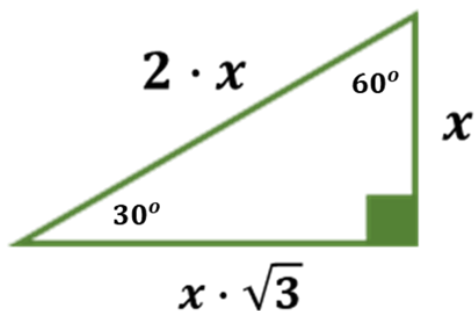


Angles of Depression and Elevation used in Word problems for Trigonometry



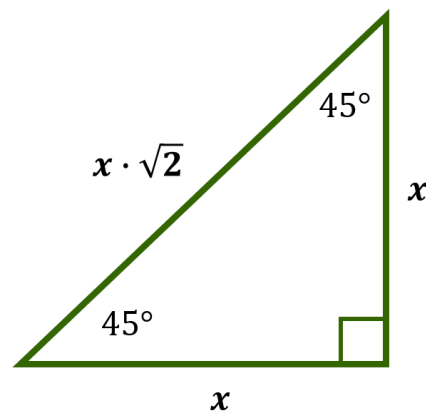
Angles of Depression and Elevation are **Alternate Interior Angles** and they are congruent.

Scalene Special Right Triangle: $30^\circ - 60^\circ - 90^\circ$



In a $30^\circ - 60^\circ - 90^\circ$ triangle, the hypotenuse is twice as long as the shorter leg $2 \cdot x$ and the longer leg is $x \cdot \sqrt{3}$ times as long as the shorter leg.

Isosceles Special Right Triangle: $45^\circ - 45^\circ - 90^\circ$



In a $45^\circ - 45^\circ - 90^\circ$ triangle, the hypotenuse is $x \cdot \sqrt{2}$ times as long as any of the congruent legs x .

Polygons: Areas

Area of a Triangle

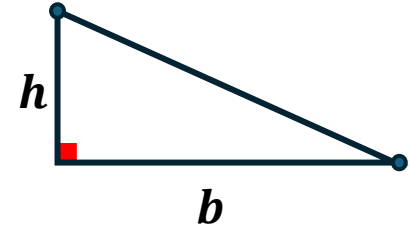
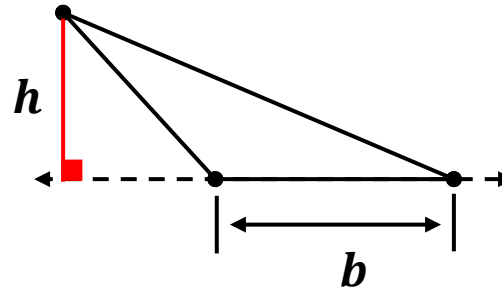
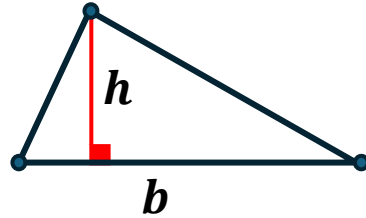
$$A = \frac{1}{2} \cdot b \cdot h$$

Where:

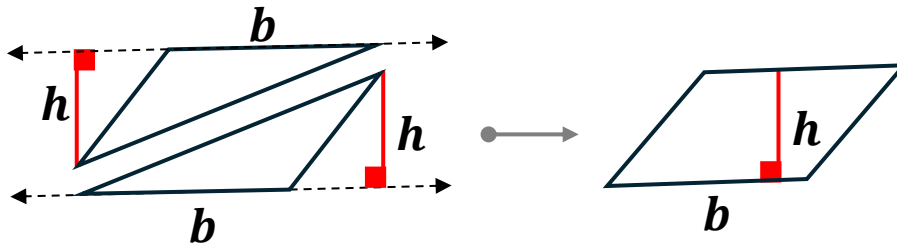
$A = \text{Area}$

$b = \text{Base}$

$h = \text{Altitude or Height}$



Area of a Parallelogram



Where:

$A = \text{Area}$

$b = \text{Base}$

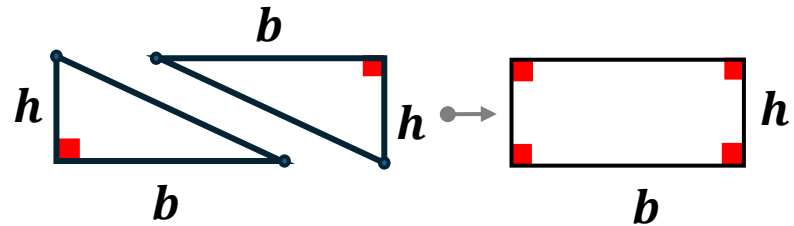
$h = \text{Altitude or Height}$

$$A = \frac{1}{2} \cdot b \cdot h + \frac{1}{2} \cdot b \cdot h$$

$$A = b \cdot h \left(\frac{1}{2} + \frac{1}{2} \right) = b \cdot h$$

$$A = b \cdot h$$

Area of a Rectangle



Where:

$A = \text{Area}$

$b = \text{Base}$

$h = \text{Altitude or Height}$

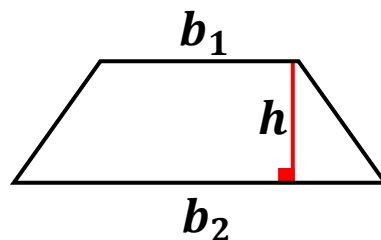
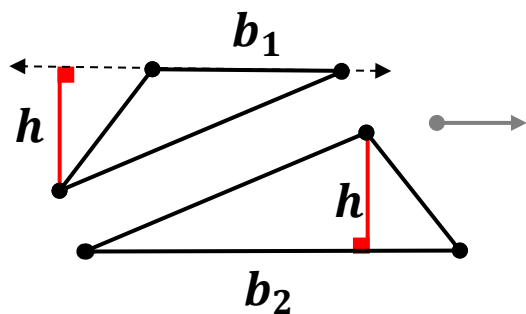
$$A = \frac{1}{2} \cdot b \cdot h + \frac{1}{2} \cdot b \cdot h$$

$$A = b \cdot h \left(\frac{1}{2} + \frac{1}{2} \right) = b \cdot h$$

$$A = b \cdot h$$

Polygons: Areas

Area of a Trapezoid



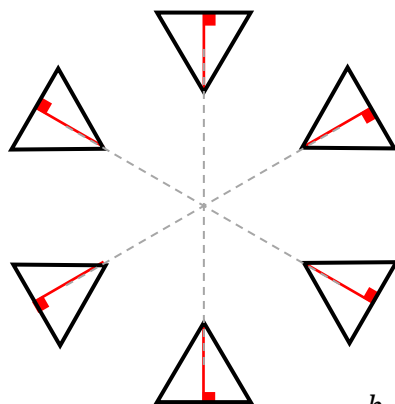
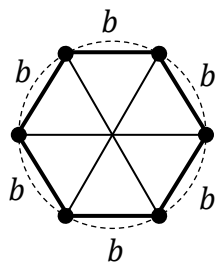
$b_1 = \text{Minor Base}$
 $b_2 = \text{Major Base}$

$A = \text{Area}$
 $h = \text{height}$
 $A = \frac{1}{2} \cdot b_1 \cdot h + \frac{1}{2} \cdot b_2 \cdot h$

$$A = \frac{1}{2} \cdot h(b_1 + b_2)$$

$M = \text{Median}$
 $M = \frac{1}{2} \cdot (b_1 + b_2)$

Area of a Regular Convex Polygon (True for all)



$$A = \frac{1}{2} \cdot hb + \frac{1}{2} \cdot hb + \frac{1}{2} \cdot hb + \frac{1}{2} \cdot hb + \frac{1}{2} \cdot hb + \frac{1}{2} \cdot hb$$

$$\rightarrow A = \frac{1}{2} \cdot h(b + b + b + b + b + b) = \frac{1}{2} \cdot P \cdot h$$

$\rightarrow \text{Equilateral } \Delta \text{ with Side "b"}$

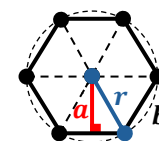
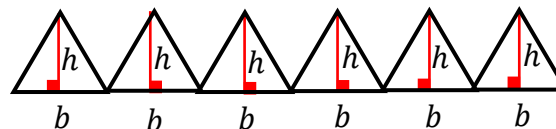
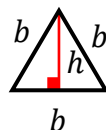
If {Regular Convex Polygon}

then { $h = \text{height} \rightarrow a = \text{Apothem}$ }

$$\text{Area} = \frac{1}{2} \cdot P \cdot a$$

Perimeter Regular Convex Polygon

$$P = b + b + b + b + b + b$$



Rhombus Area:

$$A = \frac{1}{2} \cdot (d_1 \cdot d_2)$$

Where:

$A = \text{Area}$

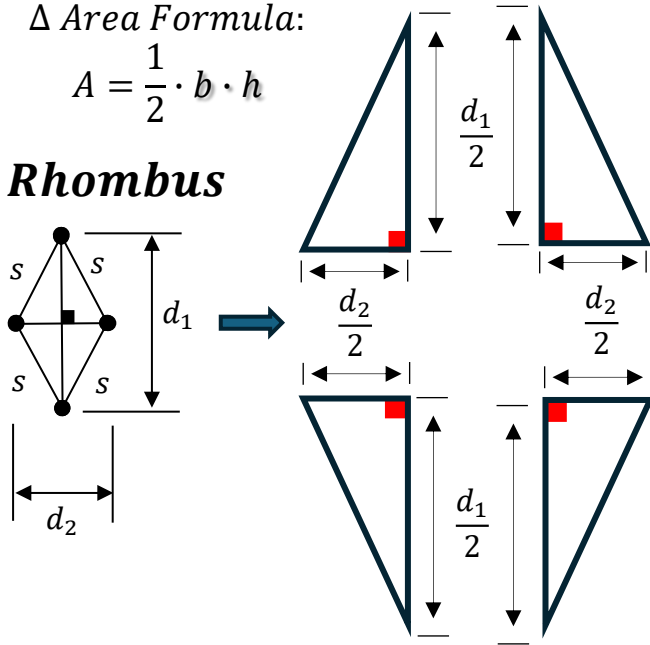
$d_1 = \text{Large Diagonal}$

$d_2 = \text{Small Diagonal}$

Δ Area Formula:

$$A = \frac{1}{2} \cdot b \cdot h$$

Rhombus



$$A = \frac{1}{2} \cdot \left(\frac{d_1}{2} \cdot \frac{d_2}{2} \right) + \frac{1}{2} \cdot \left(\frac{d_1}{2} \cdot \frac{d_2}{2} \right) + \frac{1}{2} \cdot \left(\frac{d_1}{2} \cdot \frac{d_2}{2} \right) + \frac{1}{2} \cdot \left(\frac{d_1}{2} \cdot \frac{d_2}{2} \right)$$

$$A = 4 \cdot \left[\frac{1}{2} \cdot \left(\frac{d_1}{2} \cdot \frac{d_2}{2} \right) \right]$$

$$= \frac{4}{2} \cdot \left(\frac{d_1}{2} \cdot \frac{d_2}{2} \right)$$

$$= 2 \cdot \left(\frac{d_1}{2} \cdot \frac{d_2}{2} \right) = \frac{1}{2} \cdot (d_1 \cdot d_2)$$

Area Circle:

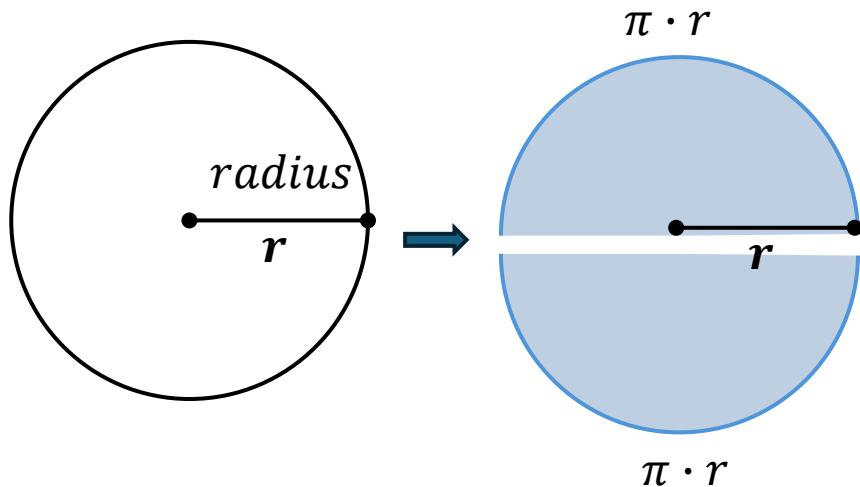
$$A = \pi \cdot r^2$$

Perimeter of a $\odot$:

$$P = 2 \cdot \pi \cdot r$$

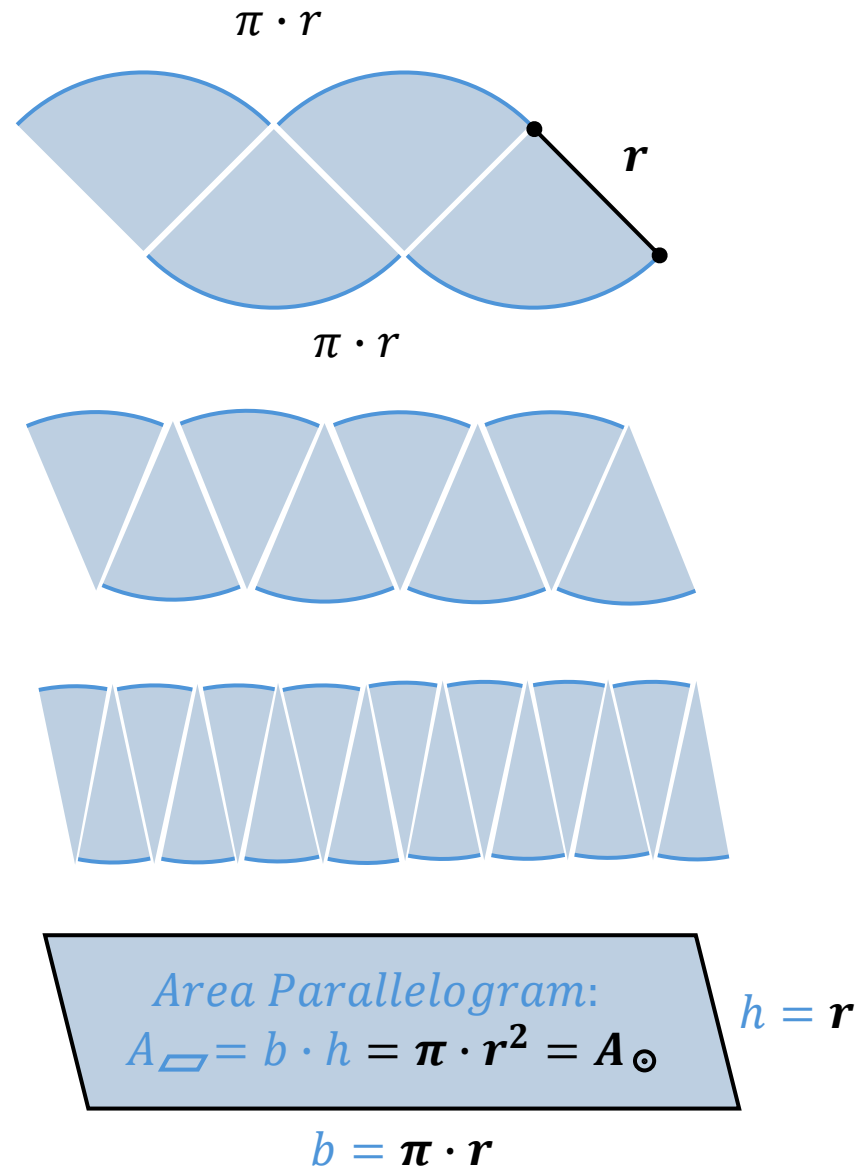
Semi - Perimeter:

$$P = \frac{\cancel{2} \cdot \pi \cdot r}{\cancel{2}} = \pi \cdot r$$



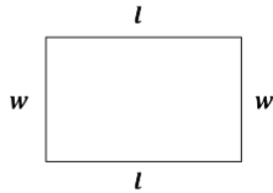
The limit of generating progressive partitions in the circle of equal area sectors is a parallelogram.

Performing successive equal area sector partitions and rearranging:



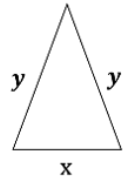
Solids: Surface Area and Volume

Perimeters Review



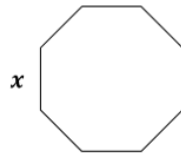
$$P = l + w + l + w$$

$$P = 2 \cdot l + 2 \cdot w$$

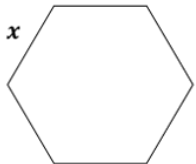


$$P = y + y + y$$

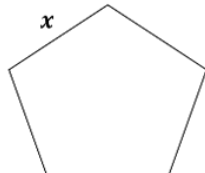
$$P = 3 \cdot y$$



$$p = 8 \cdot x$$



$$p = 6 \cdot x$$



$$p = 5 \cdot x$$

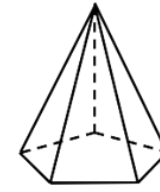


$$p = 4 \cdot x$$

Solids



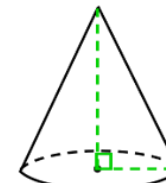
PRISM



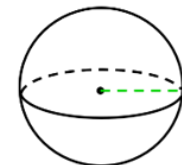
PYRAMID



CYLINDER



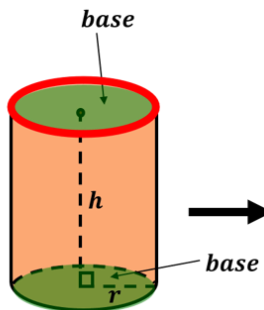
CONE



SPHERE

Surface Area in Cylinders

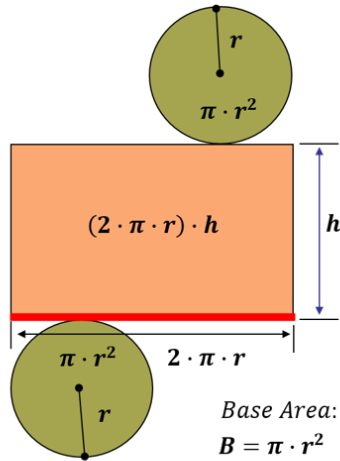
Volume in Cylinders



Lateral Area:
 $L.A. = (2 \cdot \pi \cdot r) \cdot h$
 Total Surface Area:

$$S.A. = L.A. + 2 \cdot B$$

$$S.A. = (2 \cdot \pi \cdot r) \cdot h + 2 \cdot \pi \cdot r^2$$



Base Area:
 $B = \pi \cdot r^2$
 $r = \text{radius}$
 $h = \text{height or Altitude}$

$V = \text{Volume}$

$$V = Bh$$

$$V = (\pi \cdot r^2) \cdot h$$

Base Area:

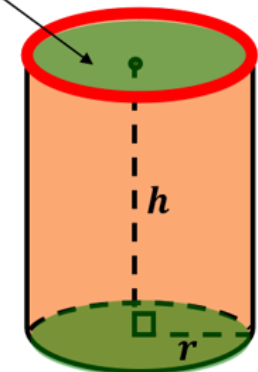
$$B = \pi \cdot r^2$$

$r = \text{radius}$

$h = \text{height or Altitude}$

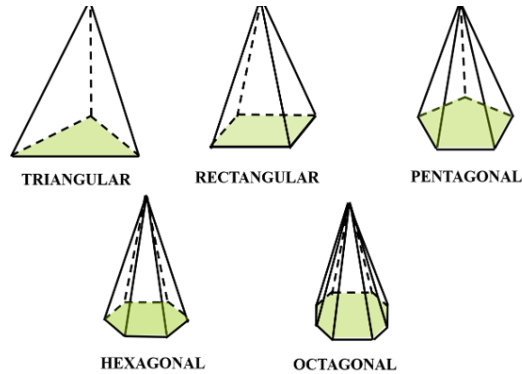
RIGHT CYLINDER

$$B = \pi \cdot r^2$$

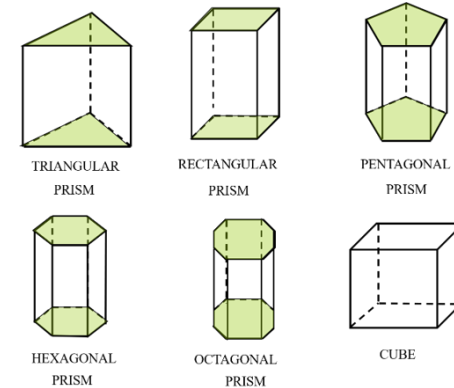


Solids: Surface Area and Volume

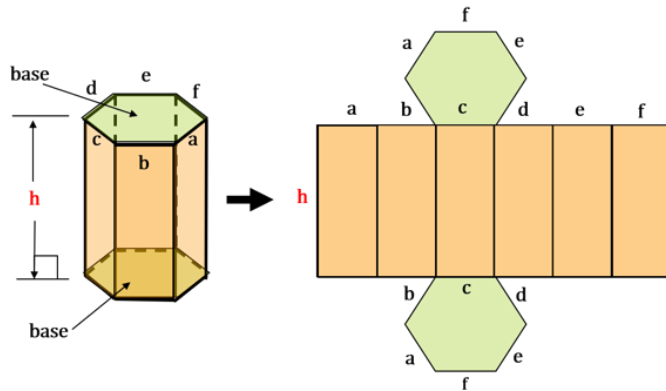
Types of Pyramids



Types of Prisms



Surface Area in Prisms



Lateral Area:

$$L = ah + bh + ch + dh + eh + fh$$

$$L = (a + b + c + d + e + f)h$$

Perimeter of base:

$$P = a + b + c + d + e + f$$

Substituting

$$L = Ph$$

Total Area = Lateral Area + 2 Base Area

$$T = L + 2B$$

$$T = Ph + 2B$$

Where:

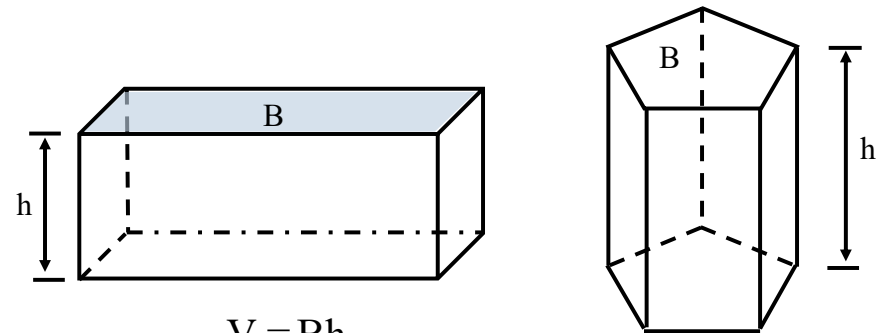
P = perimeter

h = height

B = base area

L = lateral area

Volume in a Right Prism



$$V = Bh$$

where:

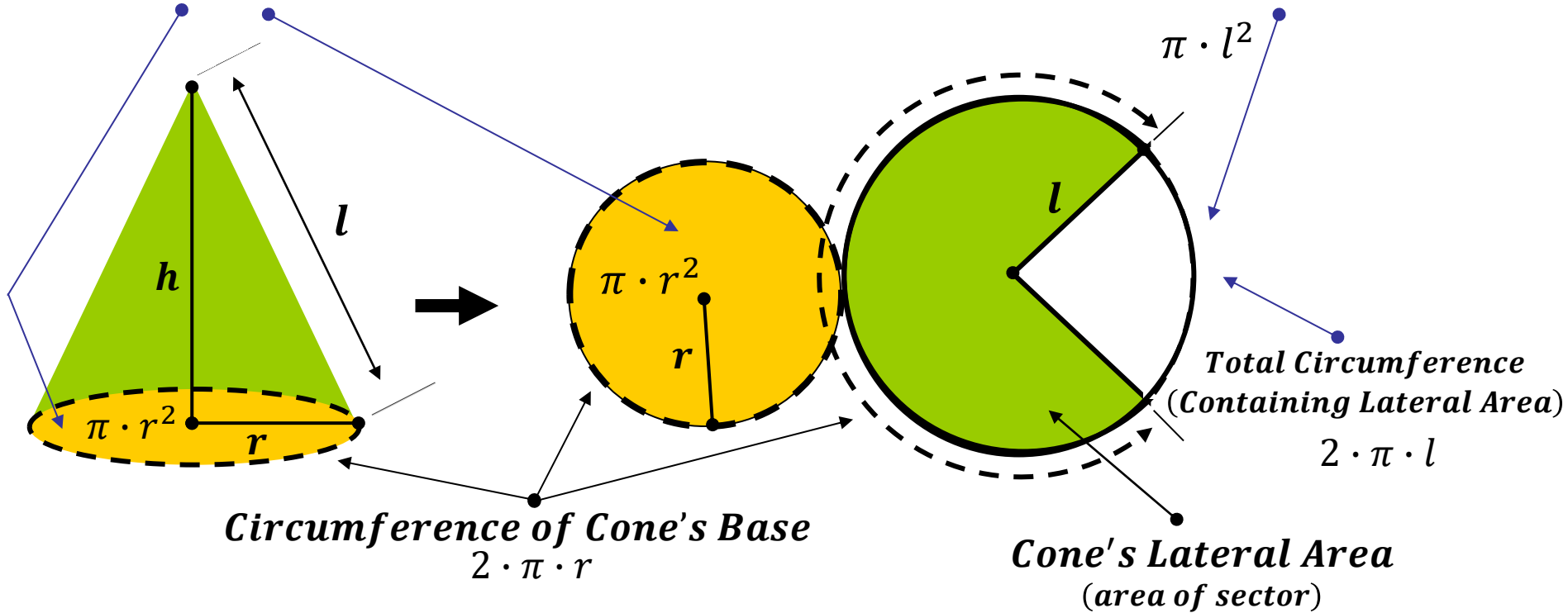
B = Area of the base

h = height

Cone's Surface Area:

Base Area Circle

Area of Circle
(Containing Lateral Area)



$$\frac{\text{Cone's Lateral Area}}{\text{Total Area Circle (Containing Lateral Area)}} = \frac{\text{Cone's Base Circumference}}{\text{Total Circumference (Containing Lateral Area)}}$$

$$\frac{\text{Cone's Lateral Area}}{\cancel{\pi \cdot l^2}} = \frac{\cancel{2 \cdot \pi \cdot r}}{\cancel{2 \cdot \pi \cdot l}}$$

$$\frac{\text{Cone's Lateral Area}}{\pi \cdot l} = r$$

$$\text{Cone's Lateral Area} = \pi \cdot r \cdot l$$

Total Surface Area:

$$S.A. = \text{Circle Base Area} + \text{Lateral Area}$$

$$S.A. = \pi \cdot r^2 + \pi \cdot r \cdot l$$

r = Radius

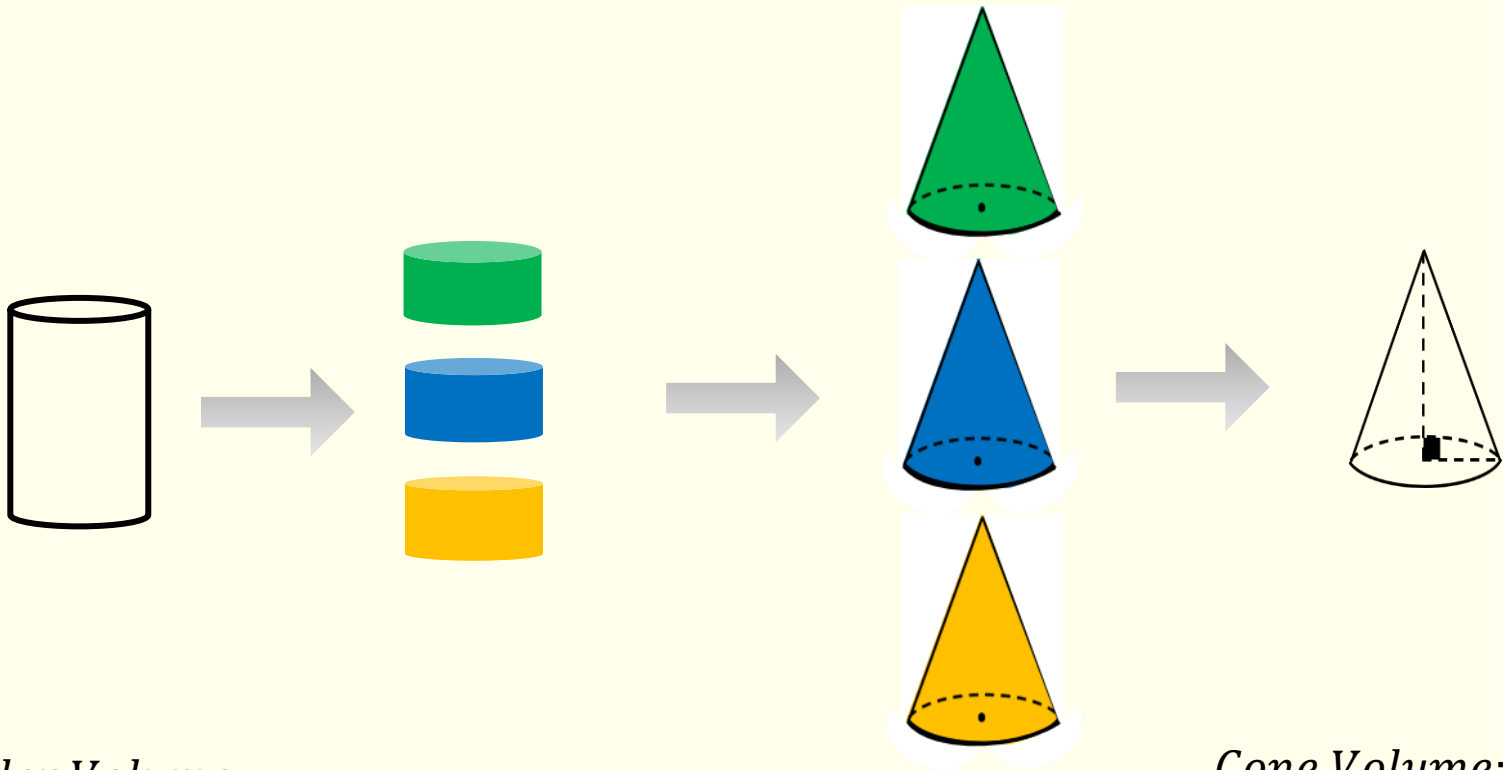
B = Circle Base Area

l = Slant Height

h = Height or Altitude

The proportion to start the proof uses the circle containing the lateral area

Comparing volume of a cylinder and a cone with the congruent bases and same height:



Cylinder Volume:

$$V = B \cdot h$$

Where:

$V = \text{Volume}$

$B = \text{Area Circle Base}$

$h = \text{height}$

*Volume of a cone
fits three times in the
volume of a cylinder with
congruent circles at
their bases and same height.*

Cone Volume:

$$V = \frac{1}{3} \cdot B \cdot h$$

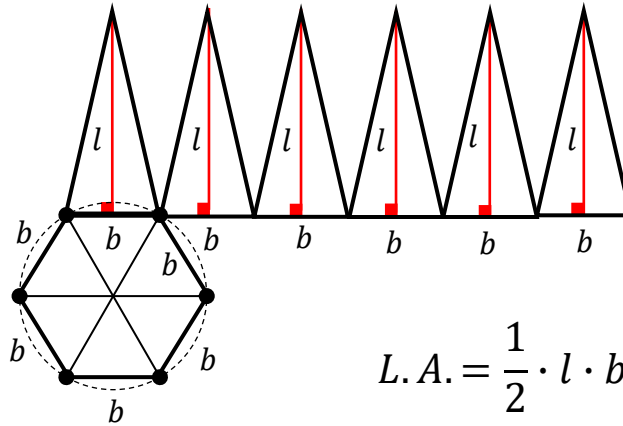
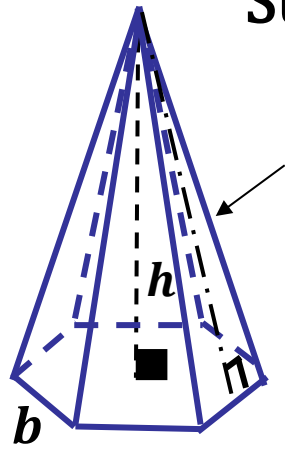
Where:

$V = \text{Volume}$

$B = \text{Area Circle Base}$

$h = \text{height}$

Surface Area in Pyramids



Lateral Area made up of Isosceles Δ 's

Perimeter_{hexagon} (convex polygon)

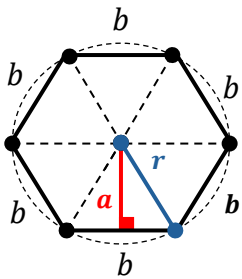
$$P = b + b + b + b + b + b$$

Lateral Area_{pyramid}:

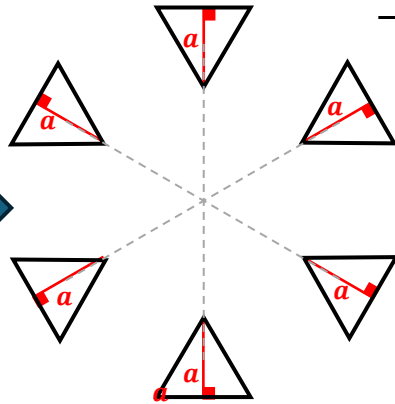
$$L.A. = \frac{1}{2} \cdot l \cdot b + \frac{1}{2} \cdot l \cdot b + \frac{1}{2} \cdot l \cdot b + \frac{1}{2} \cdot l \cdot b + \frac{1}{2} \cdot l \cdot b + \frac{1}{2} \cdot l \cdot b$$

$$\rightarrow L.A. = \frac{1}{2} \cdot l \cdot (b + b + b + b + b + b) = \frac{1}{2} \cdot P \cdot l$$

$r = \text{Radius}$



$a = \text{Apothem}$



Base Area_{hexagon}:

$$B = \frac{1}{2} \cdot a \cdot b + \frac{1}{2} \cdot a \cdot b + \frac{1}{2} \cdot a \cdot b + \frac{1}{2} \cdot a \cdot b + \frac{1}{2} \cdot a \cdot b + \frac{1}{2} \cdot a \cdot b$$

$$\rightarrow B = \frac{1}{2} \cdot a \cdot (b + b + b + b + b + b)$$

$$B = \frac{1}{2} \cdot P \cdot a$$

$$L.A. = \frac{1}{2} \cdot \text{Perimeter} \cdot \text{Slant height}$$

$$L.A. = \frac{1}{2} \cdot P \cdot l$$

$B = \text{Base Area}$

$l = \text{Slant - height}$

$h = \text{height or Altitude}$

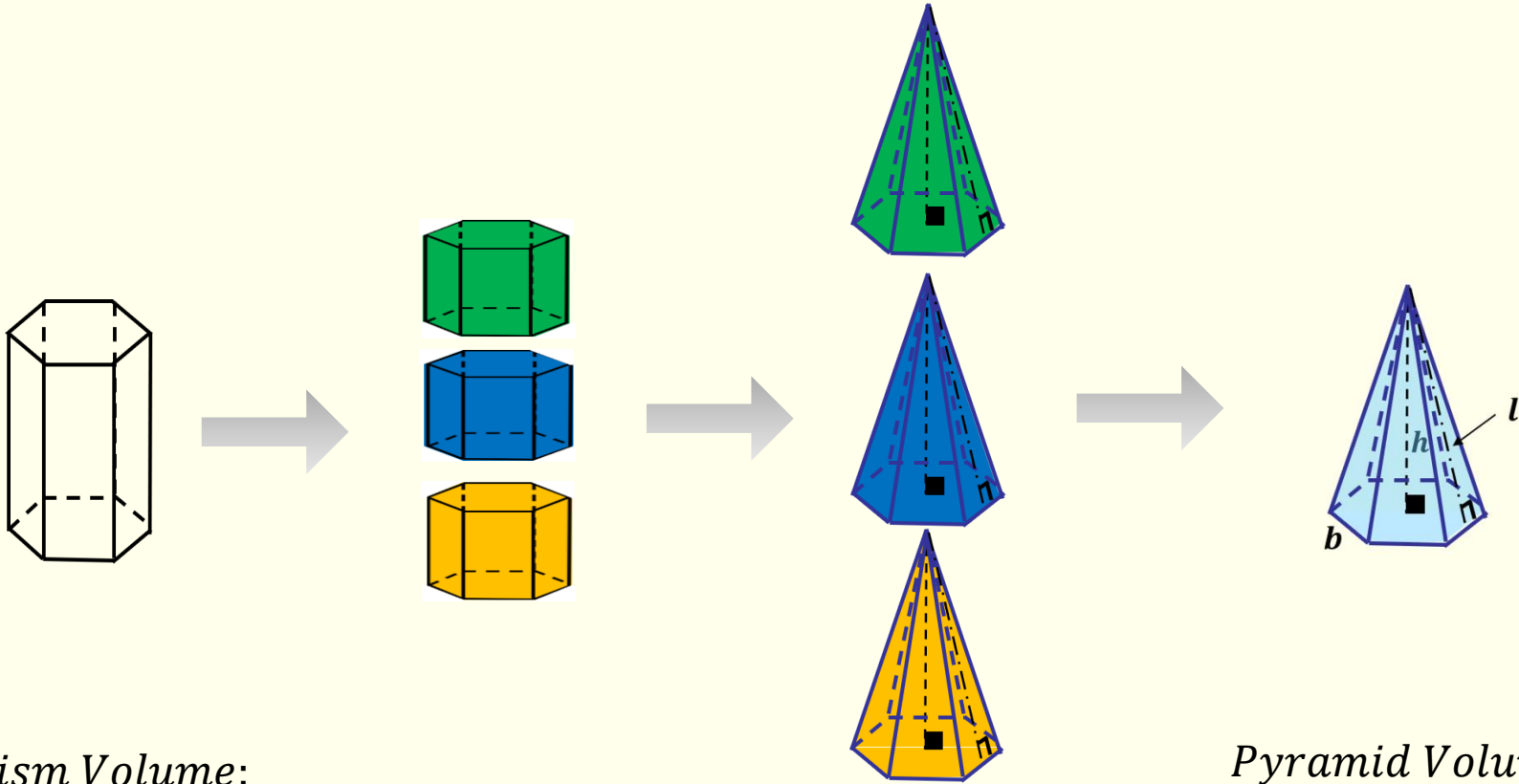
All equilateral Δ 's are isosceles but not all isosceles Δ 's are equilateral Δ 's.

Total Surface Area_{pyramid}:

$S.A. = \text{Lateral Area} + \text{Base Area}$

$$S.A. = \frac{1}{2} \cdot P \cdot l + \frac{1}{2} \cdot P \cdot a$$

Comparing volume of a prism and a pyramid with the congruent bases and same height:



Prism Volume:

$$V = B \cdot h$$

Where:

V = Volume

B = Area Base Polygon

h = height

*Volume of a pyramid
fits three times in the
volume of a prism with
congruent polygons at
their bases and same height.*

Pyramid Volume:

$$V = \frac{1}{3} \cdot B \cdot h$$

Where:

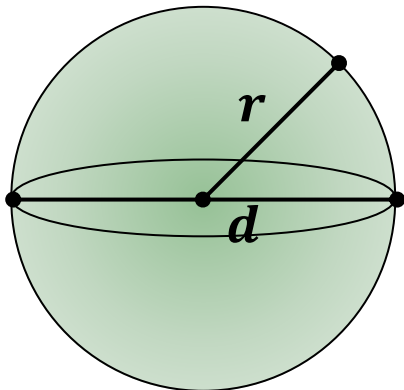
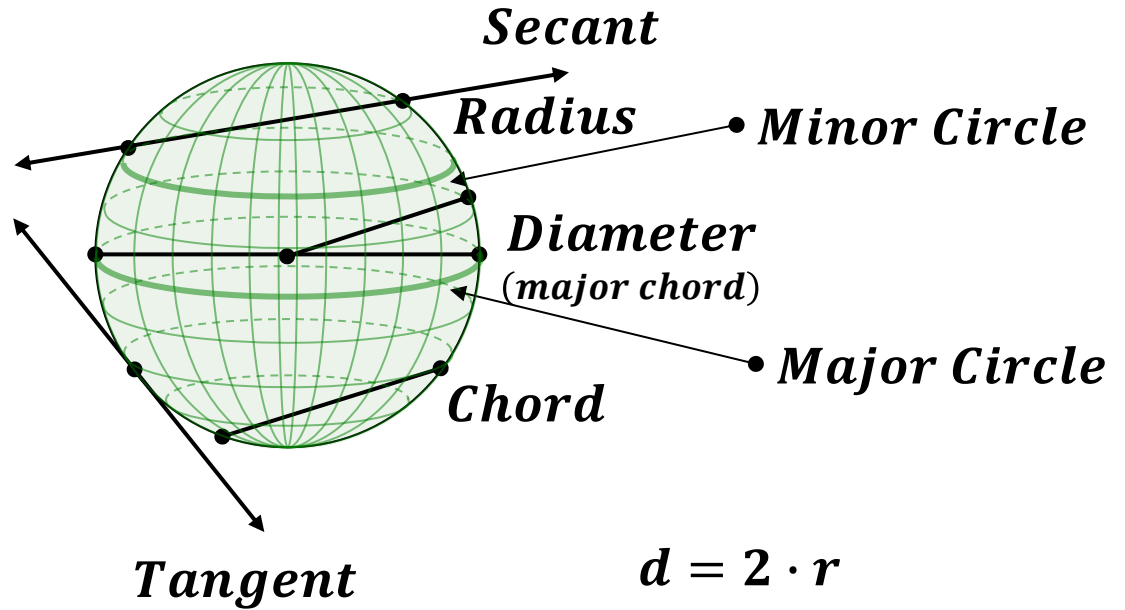
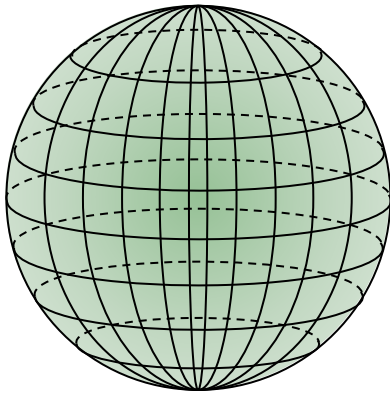
V = Volume

B = Area Base Polygon

h = height

SPHERE FEATURES

SPHERE



$$d = 2 \cdot r$$

or

$$r = \frac{1}{2} \cdot d$$

Sphere Surface Area:

$$S.A. = 4 \cdot \pi \cdot r^2$$

Sphere Volume:

$$V = \frac{4}{3} \cdot \pi \cdot r^3$$

Where:

S.A. = Surface Area

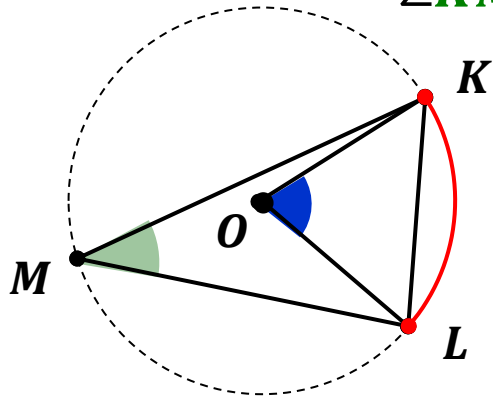
V = Volume

r = Radius

d = Diameter

Inscribed Angle vs Central Angle and Arc vs Subtended Chord to Arc

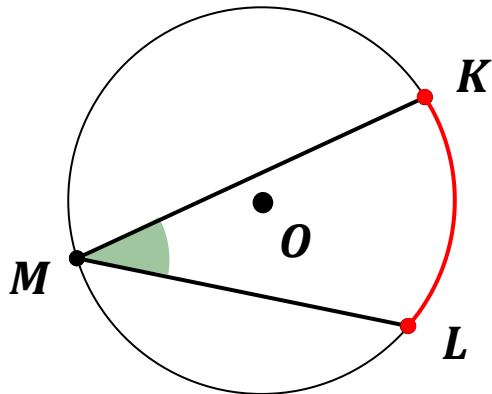
$\angle KML \rightarrow$ Inscribed Angle = $\begin{cases} \text{Vertex is on the } \odot \\ \text{Rays are chords in } \odot \end{cases}$



$\widehat{KL} \rightarrow$ Intercepted arc with subtended chord $\overline{KL}$

$\angle KOL \rightarrow$ Central Angle = $\begin{cases} \text{Vertex on center of } \odot \\ \text{Rays are Radii in } \odot \end{cases}$

$m\angle KOL = m\widehat{KL} \rightarrow \begin{cases} \text{measurement of central } \angle \text{ is equal to} \\ \text{measurement of intercepted arc} \end{cases}$



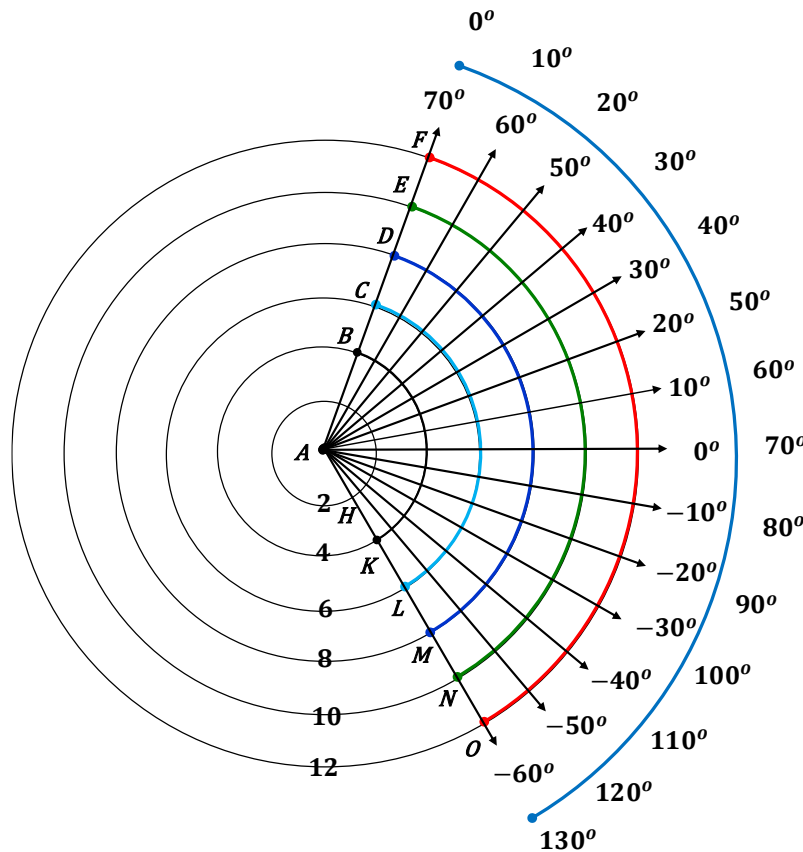
If {an angle is inscribed in a circle}

then $\begin{cases} \text{the measure of the angle equals} \\ \text{one - half the measure of its intercepted arc.} \end{cases}$

$$m\angle KLM = \frac{1}{2} m\widehat{KL}$$

Arc's Length vs Arc's Angular measurement

$$\text{Arc's Length} = 2 \cdot \pi \cdot r \cdot \left(\frac{\theta^{\circ}}{360^{\circ}} \right) \rightarrow \left\{ \begin{array}{l} \text{Arc's length is determined by the proportion} \\ \text{of its angular measurement respect} \\ \text{the total degrees in the circumference.} \\ \text{Angular units may be in Degrees or Radians} \end{array} \right\}$$



$$F \text{ --- } O$$

$$m\widehat{FO} = 130^{\circ} \rightarrow \text{Angular measurement for } \widehat{FO}$$

$$FO = 2 \cdot \pi \cdot 12 \cdot \left(\frac{130^{\circ}}{360^{\circ}} \right) \rightarrow \text{linear units for } \widehat{FO}$$

$$E \text{ --- } N$$

$$m\widehat{EN} = 130^{\circ} \rightarrow \text{Angular measurement for } \widehat{EN}$$

$$EN = 2 \cdot \pi \cdot 10 \cdot \left(\frac{130^{\circ}}{360^{\circ}} \right) \rightarrow \text{linear units for } \widehat{EN}$$

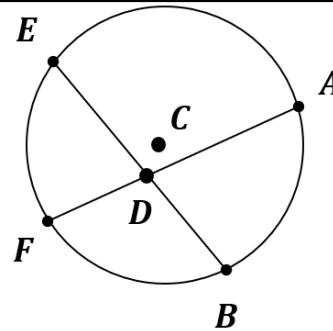
$$D \text{ --- } M$$

$$m\widehat{DM} = 130^{\circ} \rightarrow \text{Angular measurement for } \widehat{DM}$$

$$DM = 2 \cdot \pi \cdot 8 \cdot \left(\frac{130^{\circ}}{360^{\circ}} \right) \rightarrow \text{linear units for } \widehat{DM}$$

Circles: Arcs, Chords, Secants and Tangents

Two Chords Intersecting in a Circle

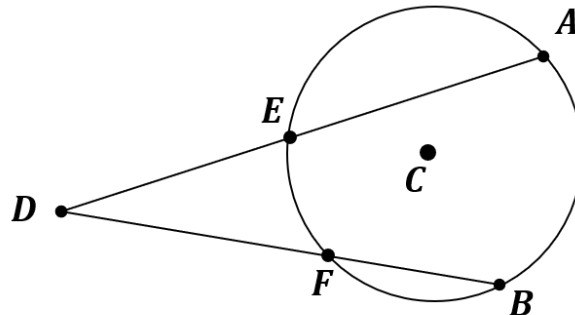


$$ED \cdot DB = FD \cdot AD$$

If {Two chords intersect in a circle}

*then {The product of their corresponding segments
formed by the intersection are equal}*

Two Secant Intersecting in a Point at the Exterior of a Circle



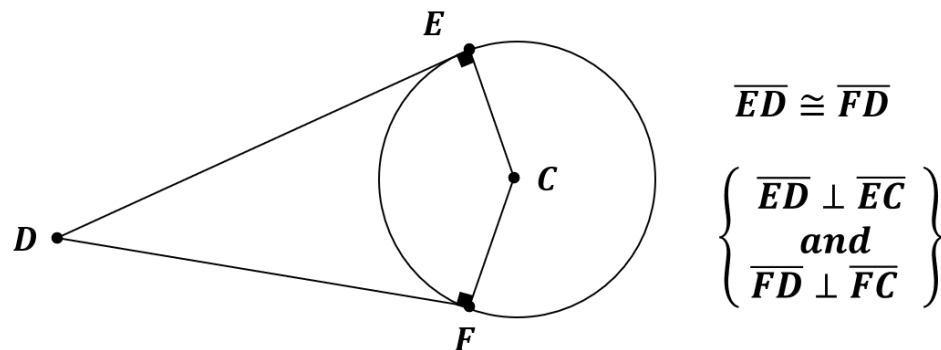
$$AD \cdot DE = BD \cdot BF$$

If {Two secants intersect in a point at the exterior of a circle}

*then {The product of the secant's length by the length of its exterior
corresponding segment is equal to the product of the other secant's
length by its corresponding exterior segment length}*

Circles: Arcs, Chords, Secants and Tangents

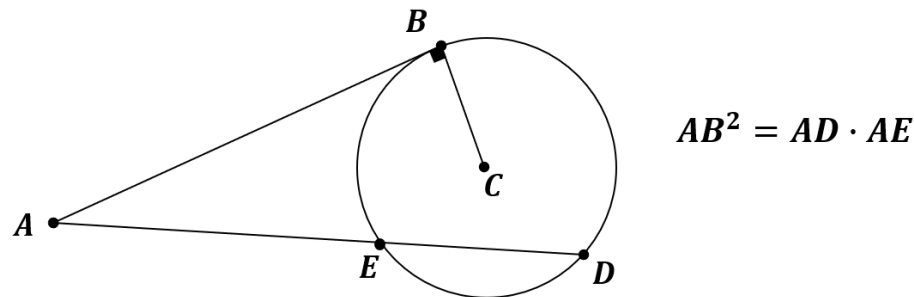
Two Tangents Intersecting in a Point at the Exterior of a Circle



If {Two tangents intersect in a point at the exterior of a circle}

then {They are $\cong$ and they are $\perp$ at the point of tangency with their corresponding radii}

A Tangent and a Secant Intersecting in a Point at the Exterior of a Circle

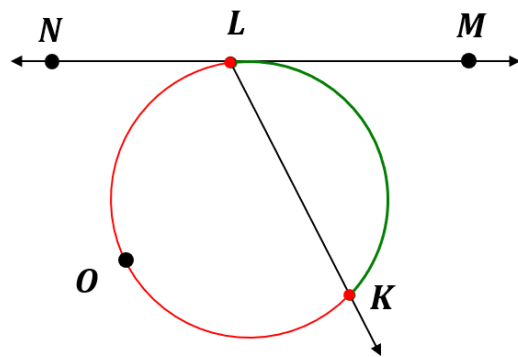


If {A tangent and a secant intersect in a point at the exterior of a circle}

then {The square of the tangent's length is equal to the product of the secant's full length by its exterior segment's length}

Circles: Arcs, Chords, Secants and Tangents

A Secant and a Tangent Intersect at the Point of Tangency



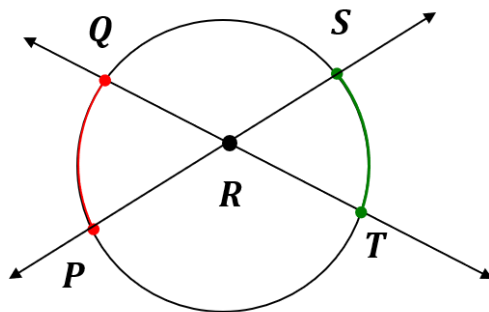
$$m\angle KLM = \frac{1}{2} m\widehat{KL}$$

$$m\angle KLN = \frac{1}{2} m\widehat{KOL}$$

If {a secant and a tangent intersect at the point of tangency}

then { the measure of each angle formed is
one – half the measure of its intercepted arc. }

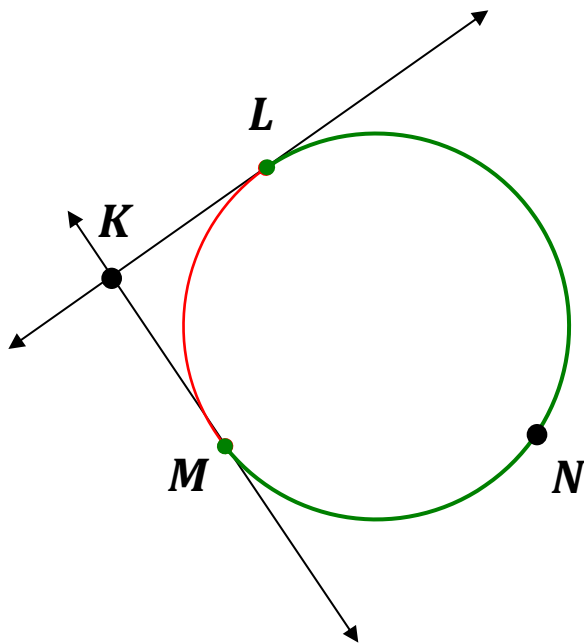
Two Secants Intersect in the Interior of a Circle



$$m\angle QRP = \frac{1}{2} (m\widehat{ST} + m\widehat{QP})$$

If {two secants intersect in the interior of a circle}

then { the measure of each angle formed is
one – half the sum of the measures of the arcs
intercepted by the angle and its vertical angle. }

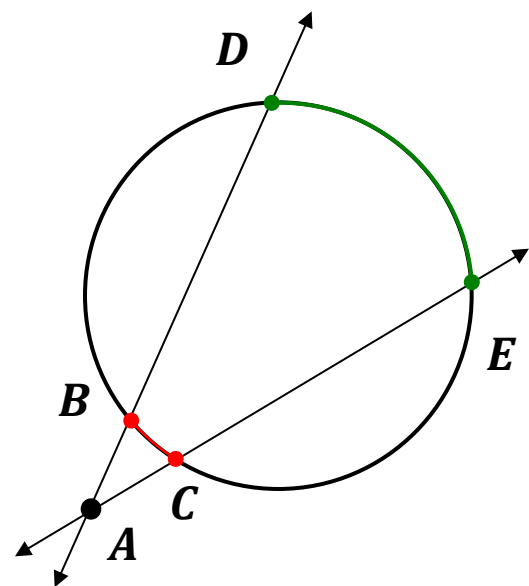
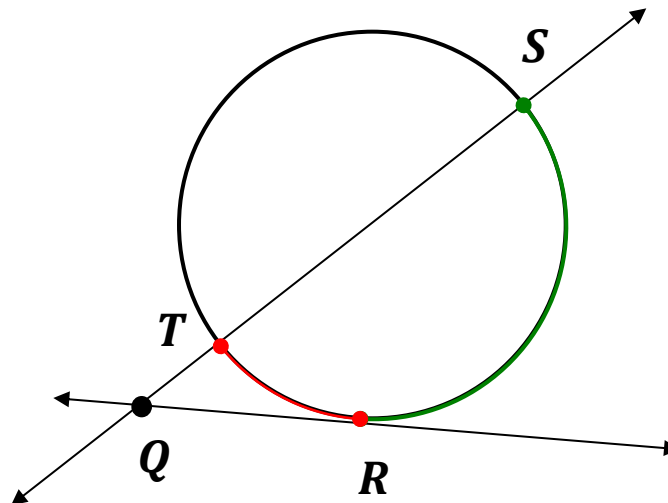


Two tangents

$$m\angle LKM = \frac{1}{2} |m\widehat{LNM} - m\widehat{LM}|$$

Secant and a tangent

$$m\angle TQR = \frac{1}{2} |m\widehat{SR} - m\widehat{TR}|$$

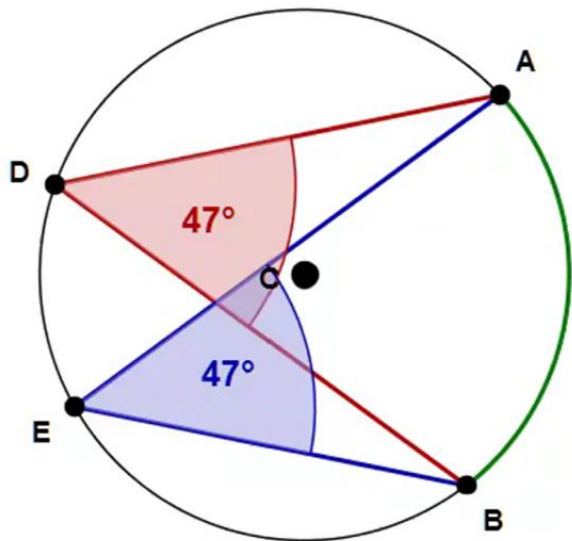


Two secants

$$m\angle BAC = \frac{1}{2} |m\widehat{DE} - m\widehat{BC}|$$

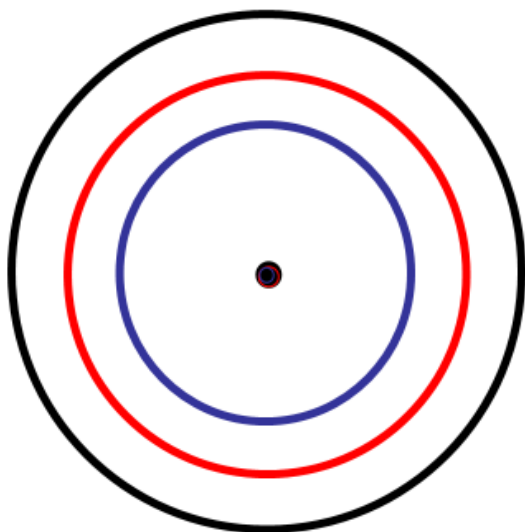
If { two secants, a secant and a tangent, or two tangents intersect in the exterior of a circle }

then { the measure of the formed angle is the absolute value of one – half the difference of the measures of the intercepted arcs or their positive semi – difference. }



*If $\{\angle ADB \text{ and } \angle BEA \text{ intercept } \widehat{AB}\}$
 then $\{\angle ADB \cong \angle BEA\}$*

*If $\left\{ \begin{array}{l} \text{Two inscribed angles of a circle or congruent} \\ \text{circles intercept congruent arcs, or the same arc} \end{array} \right\}$
 then $\{\text{The angles are congruent}\}$*



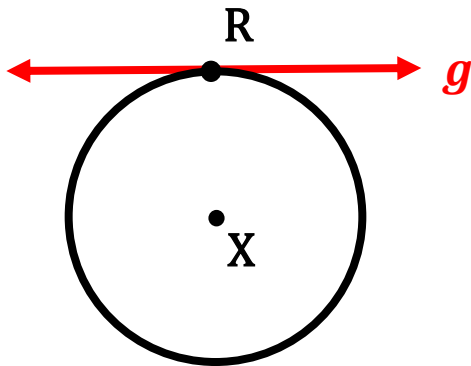
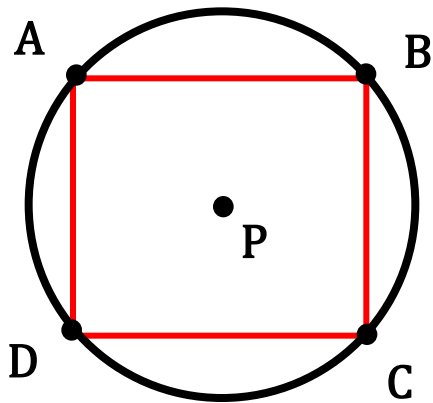
*A set of $\odot'$ s sharing the same
 center are **CONCENTRIC** and
 all $\odot'$ s are $\sim$
 since they have same shape and
 equal size or different Sizes*

$a:b$

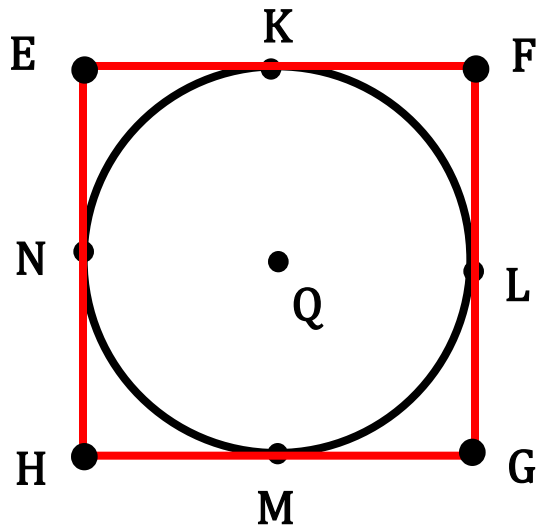
*where $a \neq b$ or $a = b$ and " a " and " b " are radii
 if $\{a = b\}$ then $\{\text{they are } \sim \text{ and } \cong\}$
 if $\{a \neq b\}$ then $\{\text{they are only } \sim \text{ and } \neq\}$*

Inscribed vs Circumscribed

Quadrilateral ABCD is **inscribed** to circle P.

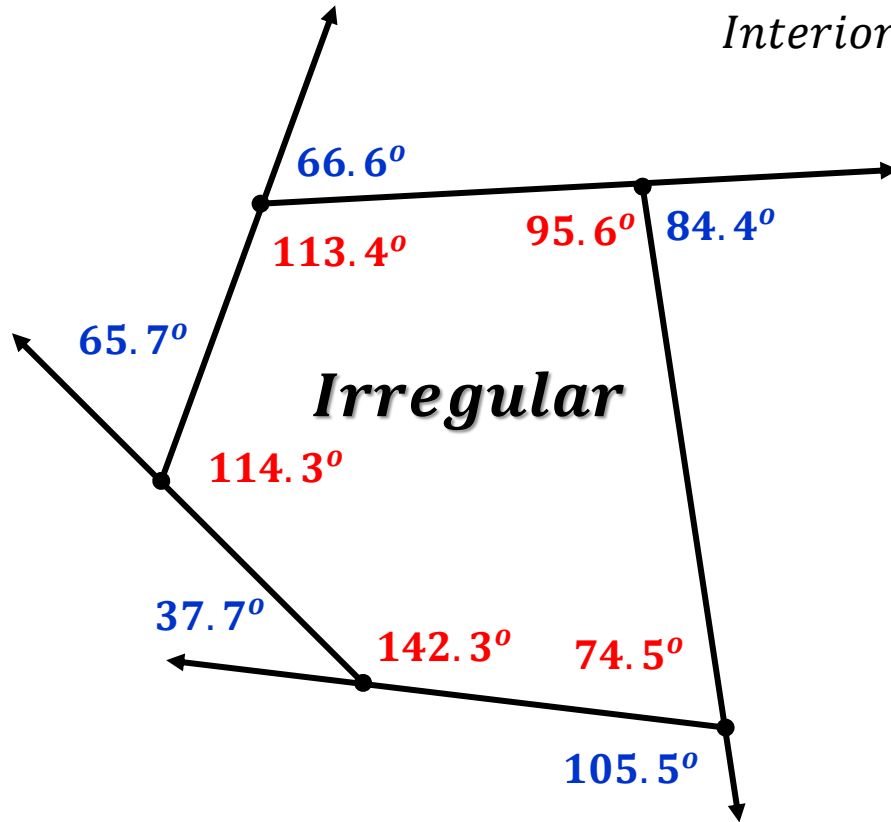


Line *g* is **TANGENT** to circle X at point R.



Quadrilateral EFGH is **circumscribed** to circle Q, having sides to be **TANGENT** at points K, L, M and N.

Irregular vs Regular (convex polygons)



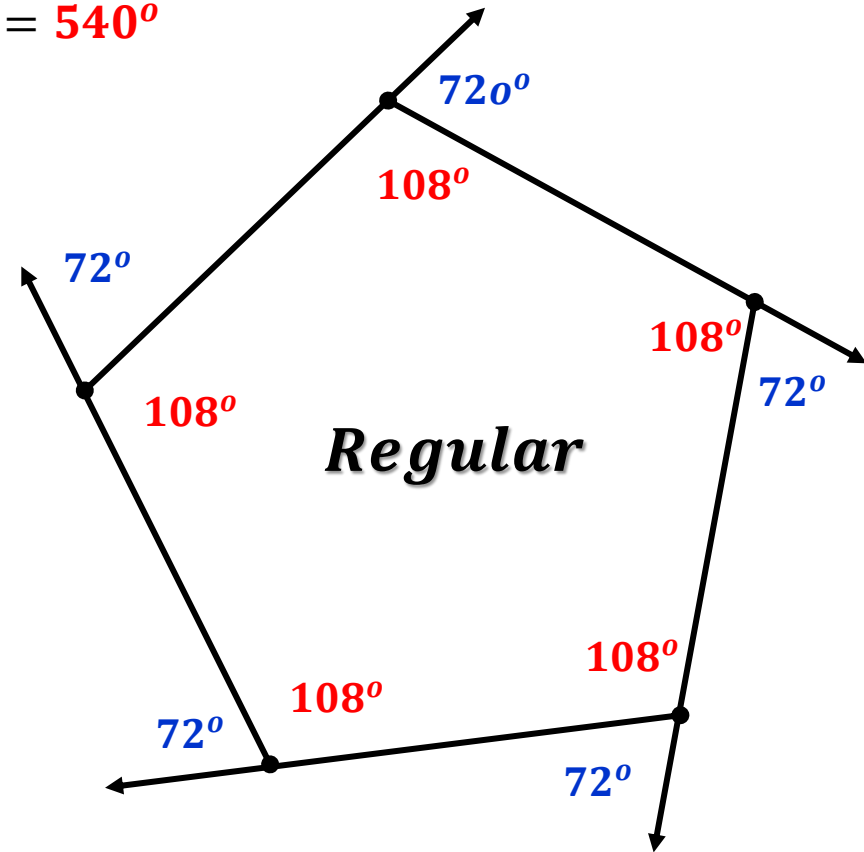
Irregular

$$\text{Exterior Sum} = 72^\circ + 72^\circ + 72^\circ + 72^\circ + 72^\circ$$

$$\text{Exterior Sum} = 360^\circ$$

$$\text{Interior Sum} = 108^\circ + 108^\circ + 108^\circ + 108^\circ + 108^\circ$$

$$\text{Interior Sum} = 540^\circ$$



Regular

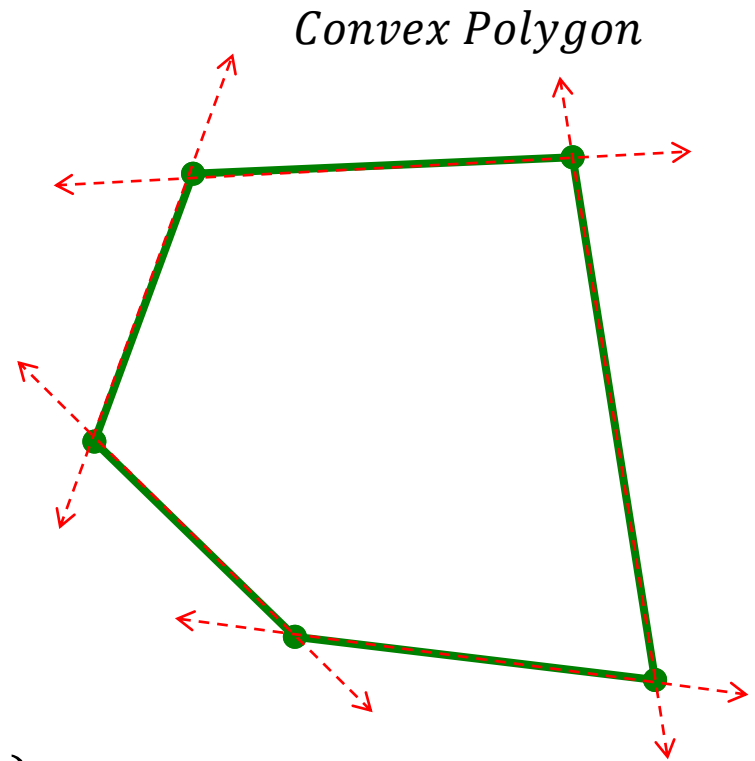
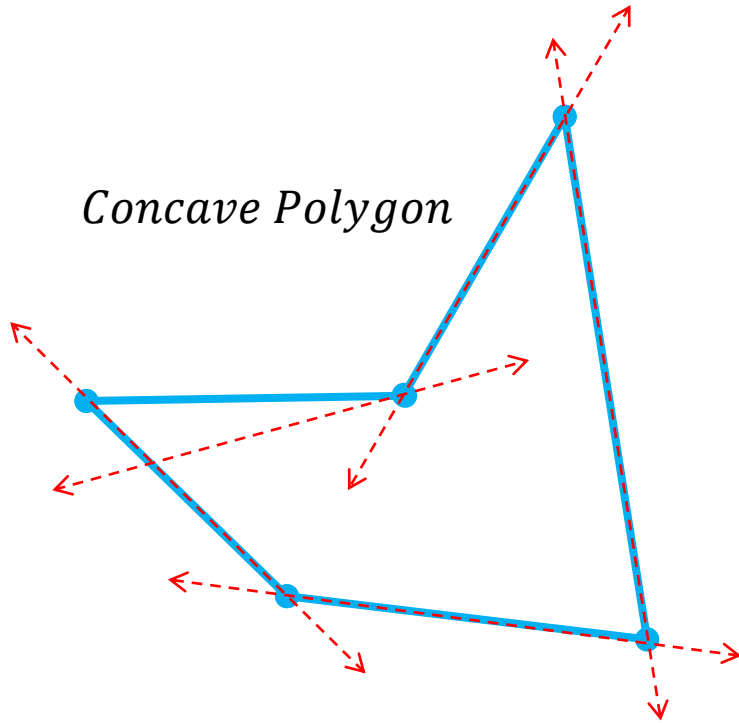
$$\text{Exterior Sum} = 65.7^\circ + 66.6^\circ + 84.4^\circ + 105.5^\circ + 37.7^\circ$$

$$\text{Exterior Sum} = 360^\circ$$

$$\text{Interior Sum} = 114.3^\circ + 113.4^\circ + 95.6^\circ + 74.5^\circ + 142.3^\circ$$

$$\text{Interior Sum} = 540^\circ$$

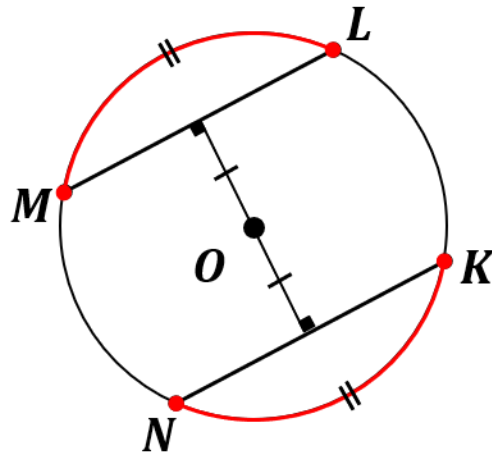
If $\left\{ \begin{array}{l} \text{Extending the sides of a polygon} \\ \text{beyond the intersection endpoints} \\ \text{of consecutive sides the **extending lines**} \\ \text{go through the interior of the polygon} \end{array} \right\}$ then {It is a **Concave Polygon**}



If $\left\{ \begin{array}{l} \text{Extending the sides of a polygon} \\ \text{beyond the intersection endpoints} \\ \text{of consecutive sides the **extending lines**} \\ \text{do not go through the interior of} \\ \text{the polygon} \end{array} \right\}$ then {It is a **Convex Polygon**}

Circles: Arcs, Chords, Secants and Tangents

Theorem about arcs and Chords and its converse



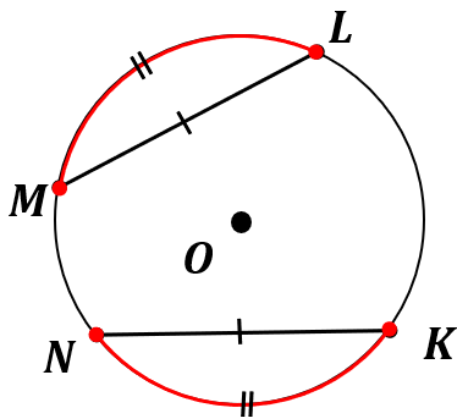
If {In a $\odot$ or in $\odot \cong'$ s two arcs are $\cong$ }

then {The arcs to which they are subtended are also $\cong$ }

If {The arcs to which they are subtended are also $\cong$ }

then {In the $\odot$ or in $\odot \cong'$ s two chords are $\cong$ }

Theorem about arcs and Chords and its converse



If {In a $\odot$ or in $\odot \cong'$ s two arcs are $\cong$ }

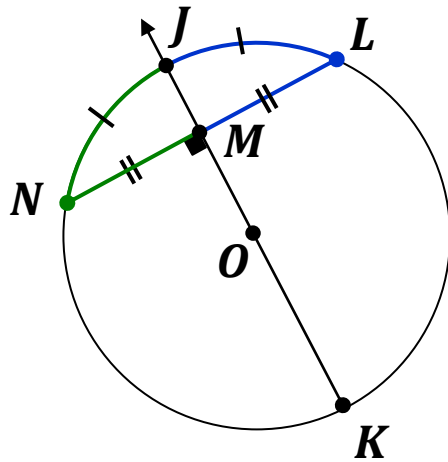
then {The subtended chords are also $\cong$ }

Si{{In a $\odot$ or in $\odot \cong'$ s two chords are $\cong$ }}

entonces {The arcs to which they are subtended are also $\cong$ }

if $\{\widehat{LM} \cong \widehat{NK}\}$ then $\{\overline{LM} \cong \overline{NK}\}$ or if $\{\overline{LM} \cong \overline{NK}\}$ then $\{\widehat{LM} \cong \widehat{NK}\}$

Theorem about a Diameter or Radius Intersecting a Chord in a Right Angle



If $\left\{ \begin{array}{l} \text{in a } \odot \text{ a chord is perpendicular} \\ \text{to a radius or diameter} \end{array} \right\}$

then $\left\{ \begin{array}{l} \text{both the chord and the arc to which} \\ \text{the chord is subtended are bisected by the} \\ \text{radius or diameter.} \end{array} \right\}$

if $\left\{ \begin{array}{l} \text{both the chord and the arc to which} \\ \text{the chord is subtended are bisected by the} \\ \text{radius or diameter.} \end{array} \right\}$

then $\left\{ \begin{array}{l} \text{in the } \odot \text{ the chord is perpendicular} \\ \text{to the radius or diameter} \end{array} \right\}$

if $\{\overline{KJ} \perp \overline{LN}\}$ *then* $\{\overline{NM} \cong \overline{LM} \text{ y } \widehat{NJ} \cong \widehat{LJ}\}$

$\overset{o}{\text{if}} \{\overline{NM} \cong \overline{LM} \text{ y } \widehat{NJ} \cong \widehat{LJ}\}$ *then* $\{\overline{KJ} \perp \overline{LN}\}$

$$2 \cdot 180^\circ = 2 \cdot \pi \text{ Radians}$$

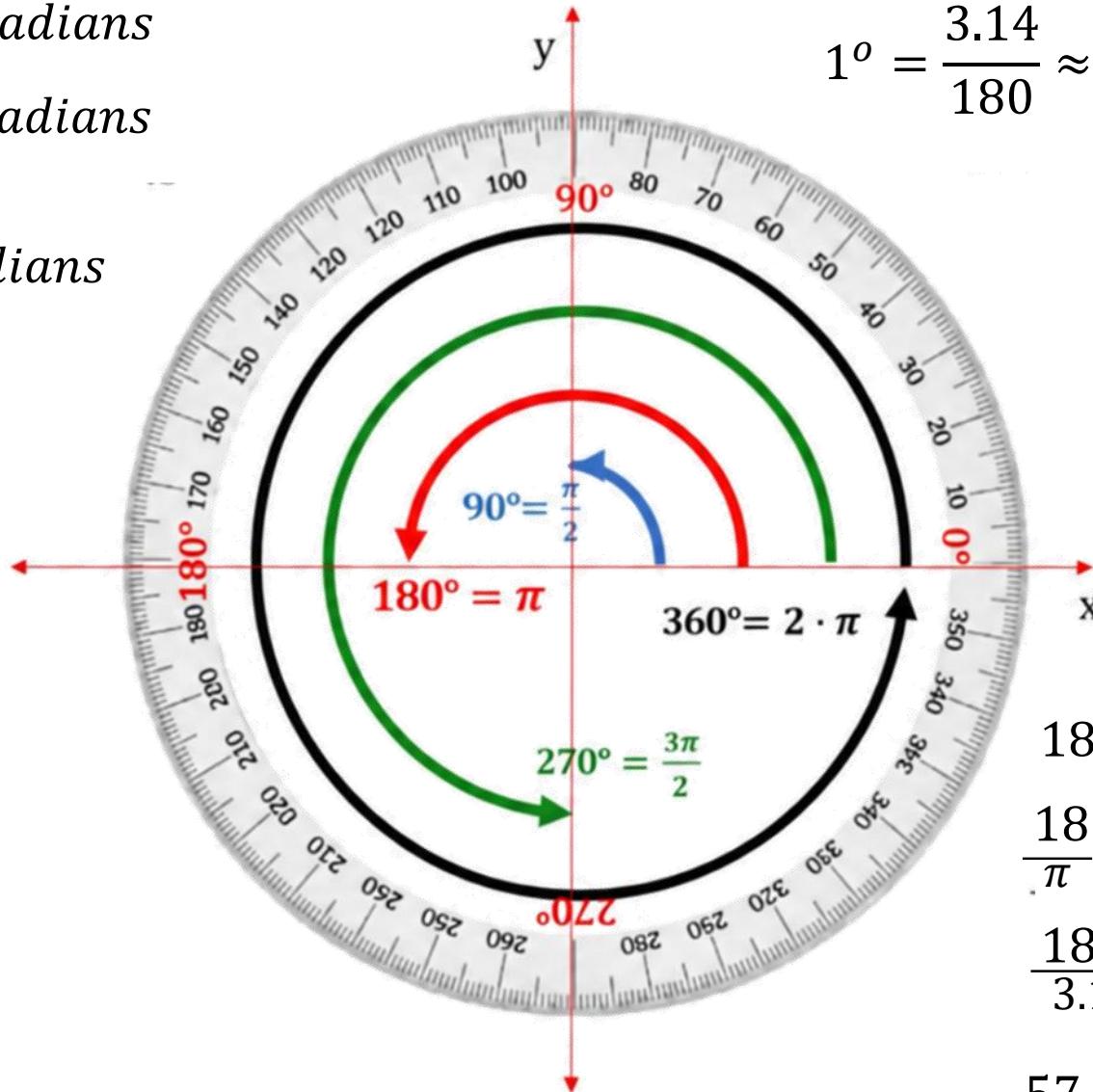
$$360^\circ = 2 \cdot \pi \text{ Radians}$$

$$\frac{360^\circ}{2} = \frac{2 \cdot \pi \text{ Radians}}{2}$$

$$180^\circ = \pi \text{ Radians}$$

$$\frac{180^\circ}{180^\circ} = \frac{\pi}{180^\circ}$$

$$1^\circ = \frac{3.14}{180} \approx 0.018 \text{ Radians}$$



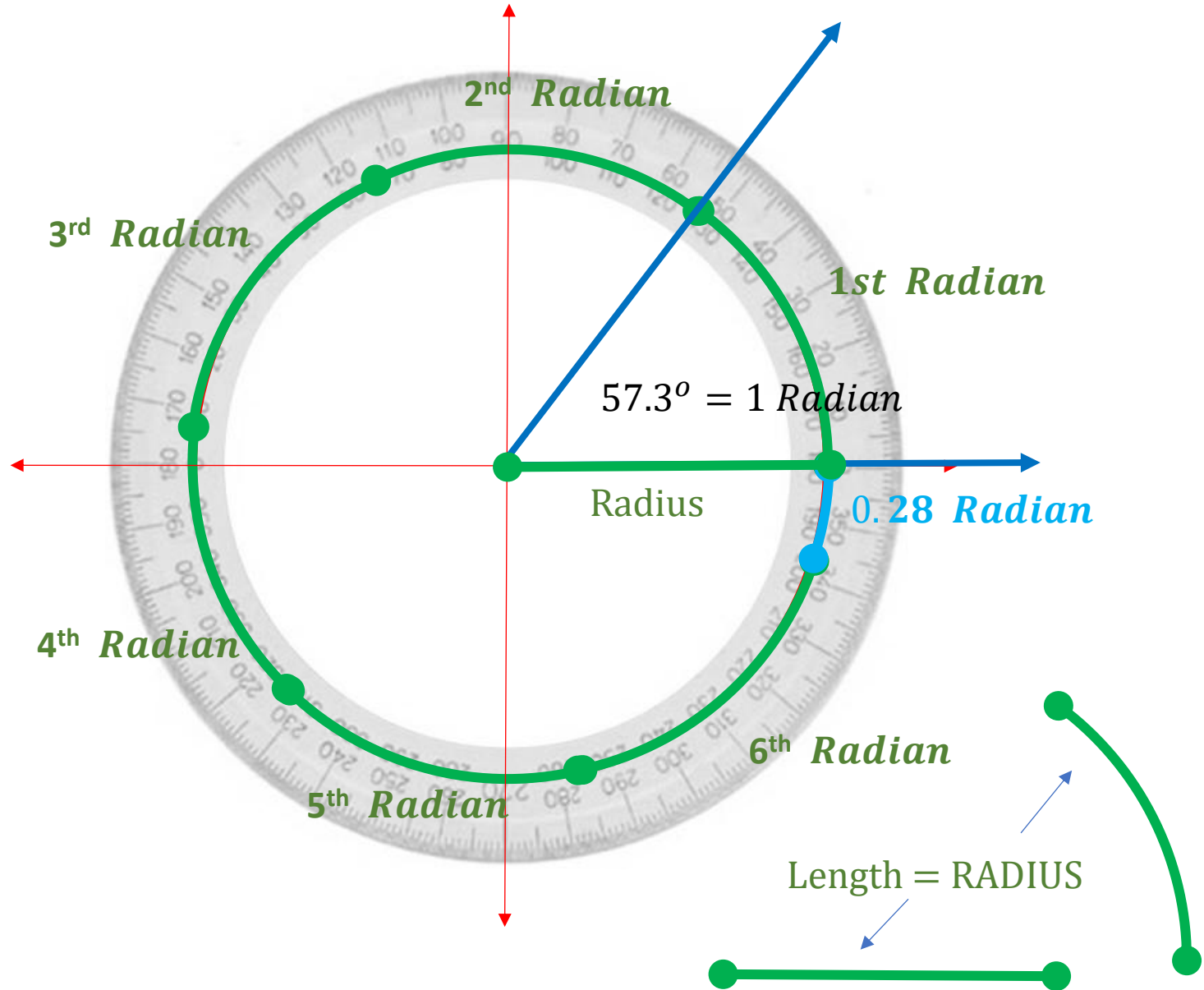
$$180^\circ = \pi$$

$$\frac{180^\circ}{\pi} = \frac{\pi}{\pi}$$

$$\frac{180^\circ}{3.14} = \frac{\pi}{\pi}$$

$$57.3^\circ = 1 \text{ Radian}$$

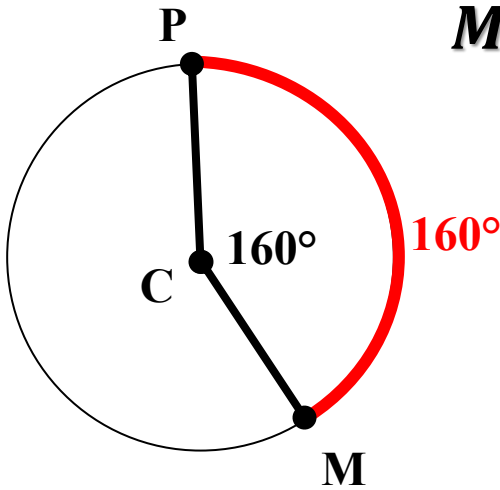
$$2 \cdot \pi = 2 \cdot 3.14 = 6.28 \text{ Radians}$$



Minor arcs, Major Arcs and Semi – Circle

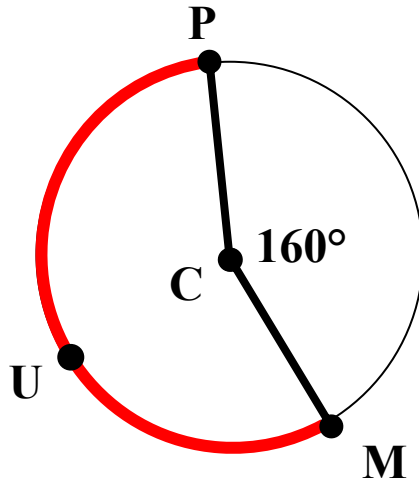
The measure of a minor arc is the measure of its central angle.

$$\angle PCM = 160^\circ \rightarrow m\widehat{PM} = 160^\circ$$



The measure of a major arc is 360° minus the measure of its central angle.

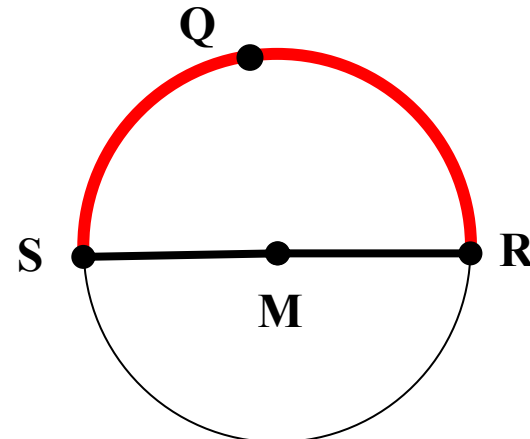
$$\angle PCM = 160^\circ \rightarrow m\widehat{PUM} = 360^\circ - 160^\circ$$



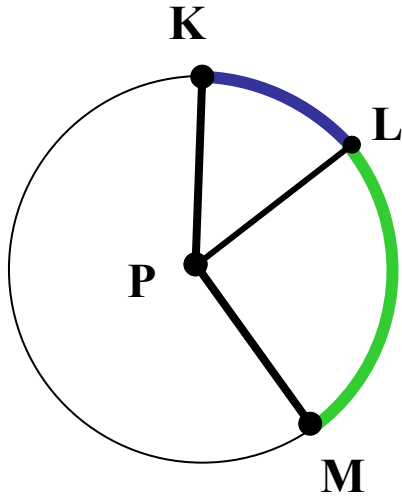
$$m\widehat{PUM} = 200^\circ$$

*The measure of a **SEMICIRCLE** is 180°*

$$m\widehat{SQR} = 180^\circ$$



Arc Addition Postulate:

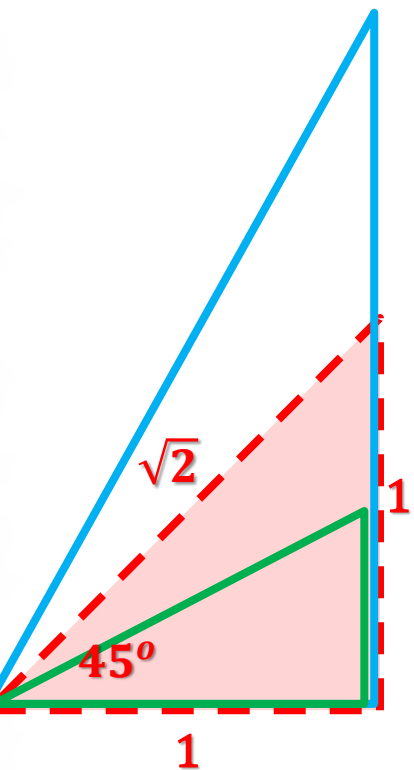


$$m\widehat{KM} = m\widehat{KL} + m\widehat{LM}$$

The measure on an arc formed by two adjacent arcs is the addition of the measures of the two arcs.

ANGLE	SINE	COSINE	TANGENT	ANGLE	SINE	COSINE	TANGENT
0	0.0000	1.0000	0.0000	46	0.7193	0.6947	1.0355
1	0.0175	0.9998	0.0175	47	0.7314	0.6820	1.0724
2	0.0349	0.9994	0.0349	48	0.7431	0.6691	1.1106
3	0.0523	0.9986	0.0524	49	0.7547	0.6561	1.1504
4	0.0698	0.9976	0.0699	50	0.7660	0.6428	1.1918
5	0.0872	0.9962	0.0875	51	0.7771	0.6293	1.2349
6	0.1045	0.9945	0.1051	52	0.7880	0.6157	1.2799
7	0.1219	0.9925	0.1228	53	0.7986	0.6018	1.3271
8	0.1392	0.9903	0.1405	54	0.8090	0.5878	1.3764
9	0.1564	0.9877	0.1584	55	0.8192	0.5736	1.4282
10	0.1736	0.9848	0.1763	56	0.8290	0.5592	1.4826
11	0.1908	0.9816	0.1944	57	0.8387	0.5446	1.5399
12	0.2079	0.9781	0.2126	58	0.8480	0.5299	1.6003
13	0.2250	0.9744	0.2309	59	0.8572	0.5150	1.6643
14	0.2419	0.9703	0.2493	60	0.8660	0.5000	1.7321
15	0.2588	0.9659	0.2679	61	0.8746	0.4848	1.8041
16	0.2756	0.9613	0.2867	62	0.8829	0.4695	1.8807
17	0.2924	0.9663	0.3057	63	0.8910	0.4540	1.9626
18	0.3090	0.9511	0.3249	64	0.8988	0.4384	2.0503
19	0.3256	0.9455	0.3443	65	0.9063	0.4226	2.1445
20	0.3420	0.9397	0.3640	66	0.9135	0.4067	2.2461
21	0.3584	0.9336	0.3839	67	0.9205	0.3907	2.3559
22	0.3746	0.9272	0.4040	68	0.9272	0.3746	2.4751
23	0.3907	0.9205	0.4245	69	0.9336	0.3584	2.6051
24	0.4067	0.9135	0.4452	70	0.9397	0.3420	2.7475
25	0.4226	0.9063	0.4663	71	0.9455	0.3256	2.9042
26	0.4384	0.8988	0.4877	72	0.9511	0.3090	3.0777
27	0.4540	0.8910	0.5095	73	0.9563	0.2924	3.2709
28	0.4695	0.8829	0.5317	74	0.9613	0.2756	3.4875
29	0.4848	0.8746	0.5543	75	0.9659	0.2588	3.7321
30	0.5000	0.8660	0.5774	76	0.9703	0.2419	4.0108
31	0.5150	0.8572	0.6009	77	0.9744	0.2249	4.3315
32	0.5299	0.8480	0.6249	78	0.9781	0.2079	4.7047
33	0.5446	0.8387	0.6494	79	0.9816	0.1908	5.1446
34	0.5592	0.8290	0.6745	80	0.9848	0.1736	5.6714
35	0.5736	0.8192	0.7002	81	0.9877	0.1564	6.3139
36	0.5878	0.8090	0.7265	82	0.9903	0.1392	7.1155
37	0.6018	0.7986	0.7536	83	0.9925	0.1219	8.1446
38	0.6157	0.7880	0.7813	84	0.9945	0.1045	9.5147
39	0.6293	0.7771	0.8098	85	0.9962	0.0872	11.4305
40	0.6428	0.7660	0.8391	86	0.9976	0.0698	14.3014
41	0.6561	0.7547	0.8693	87	0.9986	0.0523	19.0824
42	0.6692	0.7431	0.9004	88	0.9994	0.0349	28.6392
43	0.6820	0.7311	0.9325	89	0.9998	0.0174	57.3010
44	0.6947	0.7193	0.9657				
45	0.7071	0.7071	1.0000	90	1.0000	0.0000	*

Trigonometric values from tables first quadrant



ANGLE	SINE	COSINE	TANGENT	ANGLE	SINE	COSINE	TANGENT
0	0.0000	1.0000	0.0000	46	0.7193	0.6947	1.0355
1	0.0175	0.9998	0.0175	47	0.7314	0.6820	1.0724
2	0.0349	0.9994	0.0349	48	0.7431	0.6691	1.1106
3	0.0523	0.9986	0.0524	49	0.7547	0.6561	1.1504
4	0.0698	0.9976	0.0699	50	0.7660	0.6428	1.1918
5	0.0872	0.9962	0.0875	51	0.7771	0.6293	1.2349
6	0.1045	0.9945	0.1051	52	0.7880	0.6157	1.2799
7	0.1219	0.9925	0.1228	53	0.7986	0.6018	1.3271
8	0.1392	0.9903	0.1405	54	0.8090	0.5878	1.3764
9	0.1564	0.9877	0.1584	55	0.8192	0.5736	1.4282
10	0.1736	0.9848	0.1763	56	0.8290	0.5592	1.4826
11	0.1908	0.9816	0.1944	57	0.8387	0.5446	1.5399
12	0.2079	0.9781	0.2126	58	0.8480	0.5299	1.6003
13	0.2250	0.9744	0.2309	59	0.8572	0.5150	1.6643
14	0.2419	0.9703	0.2493	60	0.8660	0.5000	1.7321
15	0.2588	0.9659	0.2679	61	0.8746	0.4848	1.8041
16	0.2756	0.9613	0.2867	62	0.8829	0.4695	1.8807
17	0.2924	0.9563	0.3057	63	0.8910	0.4540	1.9626
18	0.3090	0.9511	0.3249	64	0.8988	0.4384	2.0503
19	0.3256	0.9455	0.3443	65	0.9063	0.4226	2.1445
20	0.3420	0.9397	0.3640	66	0.9135	0.4067	2.2461
21	0.3584	0.9336	0.3839	67	0.9205	0.3907	2.3559
22	0.3746	0.9272	0.4040	68	0.9272	0.3746	2.4751
23	0.3907	0.9205	0.4245	69	0.9336	0.3584	2.6051
24	0.4067	0.9135	0.4452	70	0.9397	0.3420	2.7475
25	0.4226	0.9063	0.4663	71	0.9455	0.3256	2.9042
26	0.4384	0.8988	0.4877	72	0.9511	0.3090	3.0777
27	0.4540	0.8910	0.5095	73	0.9563	0.2924	3.2709
28	0.4695	0.8829	0.5317	74	0.9613	0.2756	3.4875
29	0.4848	0.8746	0.5543	75	0.9659	0.2588	3.7321
30	0.5000	0.8660	0.5774	76	0.9703	0.2419	4.0108
31	0.5150	0.8572	0.6009	77	0.9744	0.2249	4.3315
32	0.5299	0.8480	0.6249	78	0.9781	0.2079	4.7047
33	0.5446	0.8387	0.6494	79	0.9816	0.1908	5.1446
34	0.5592	0.8290	0.6745	80	0.9848	0.1736	5.6714
35	0.5736	0.8192	0.7002	81	0.9877	0.1564	6.3139
36	0.5878	0.8090	0.7265	82	0.9903	0.1392	7.1155
37	0.6018	0.7986	0.7536	83	0.9925	0.1219	8.1446
38	0.6157	0.7880	0.7813	84	0.9945	0.1045	9.5147
39	0.6293	0.7771	0.8098	85	0.9962	0.0872	11.4305
40	0.6428	0.7660	0.8391	86	0.9976	0.0698	14.3014
41	0.6561	0.7547	0.8693	87	0.9986	0.0523	19.0824
42	0.6691	0.7431	0.9004	88	0.9994	0.0349	28.6392
43	0.6820	0.7314	0.9325	89	0.9998	0.0174	57.3019
44	0.6947	0.7193	0.9657	90	1.0000	0.0000	*
45	0.7071	0.7071	1.0000				

Angle to Ratio:

$$\cos 37^\circ = 0.7986$$

$$\sin 19^\circ = 0.3256$$

$$\tan 67^\circ = 2.3559$$

ANGLE	SINE	COSINE	TANGENT	ANGLE	SINE	COSINE	TANGENT
0	0.0000	1.0000	0.0000	46	0.7193	0.6947	1.0355
1	0.0175	0.9998	0.0175	47	0.7314	0.6820	1.0724
2	0.0349	0.9994	0.0349	48	0.7431	0.6691	1.1106
3	0.0524	0.9986	0.0524	49	0.7547	0.6561	1.1504
4	0.0698	0.9976	0.0699	50	0.7660	0.6428	1.1918
5	0.0872	0.9962	0.0875	51	0.7771	0.6293	1.2349
6	0.1045	0.9945	0.1051	52	0.7880	0.6157	1.2799
7	0.1219	0.9925	0.1228	53	0.7986	0.6018	1.3271
8	0.1392	0.9903	0.1405	54	0.8090	0.5878	1.3764
9	0.1564	0.9877	0.1584	55	0.8192	0.5736	1.4282
10	0.1736	0.9848	0.1763	56	0.8290	0.5592	1.4826
11	0.1908	0.9816	0.1944	57	0.8387	0.5446	1.5399
12	0.2079	0.9781	0.2126	58	0.8480	0.5299	1.6003
13	0.2250	0.9744	0.2309	59	0.8572	0.5150	1.6643
14	0.2419	0.9703	0.2493	60	0.8660	0.5000	1.7321
15	0.2588	0.9659	0.2679	61	0.8746	0.4848	1.8041
16	0.2756	0.9613	0.2867	62	0.8829	0.4695	1.8807
17	0.2924	0.9563	0.3057	63	0.8910	0.4540	1.9626
18	0.3090	0.9511	0.3249	64	0.8988	0.4384	2.0503
19	0.3256	0.9455	0.3443	65	0.9063	0.4226	2.1445
20	0.3420	0.9397	0.3640	66	0.9135	0.4067	2.2461
21	0.3584	0.9336	0.3839	67	0.9205	0.3907	2.3559
22	0.3746	0.9272	0.4040	68	0.9272	0.3746	2.4751
23	0.3907	0.9205	0.4245	69	0.9336	0.3584	2.6051
24	0.4067	0.9135	0.4452	70	0.9397	0.3420	2.7475
25	0.4226	0.9063	0.4663	71	0.9455	0.3256	2.9042
26	0.4384	0.8988	0.4877	72	0.9511	0.3090	3.0777
27	0.4540	0.8910	0.5095	73	0.9563	0.2924	3.2709
28	0.4695	0.8829	0.5317	74	0.9613	0.2756	3.4875
29	0.4848	0.8746	0.5543	75	0.9659	0.2588	3.7321
30	0.5000	0.8660	0.5774	76	0.9703	0.2419	4.0108
31	0.5150	0.8572	0.6009	77	0.9744	0.2249	4.3315
32	0.5299	0.8480	0.6249	78	0.9781	0.2079	4.7047
33	0.5446	0.8387	0.6494	79	0.9816	0.1908	5.1446
34	0.5592	0.8290	0.6745	80	0.9848	0.1736	5.6714
35	0.5736	0.8192	0.7002	81	0.9877	0.1564	6.3139
36	0.5878	0.8090	0.7265	82	0.9903	0.1392	7.1755
37	0.6018	0.7986	0.7536	83	0.9925	0.1219	8.1446
38	0.6157	0.7880	0.7813	84	0.9945	0.1045	9.5147
39	0.6293	0.7771	0.8098	85	0.9962	0.0872	11.4305
40	0.6428	0.7660	0.8391	86	0.9976	0.0698	14.3014
41	0.6561	0.7547	0.8693	87	0.9986	0.0523	19.0824
42	0.6691	0.7431	0.9004	88	0.9994	0.0349	28.6392
43	0.6820	0.7314	0.9325	89	0.9998	0.0174	57.3019
44	0.6947	0.7193	0.9657	90	1.0000	0.0000	*
45	0.7071	0.7071	1.0000				

Ratio to Angle:

$$\cos^{-1}(0.9205) = 23^\circ$$

$$\sin^{-1}(0.0872) = 5^\circ$$

$$\tan^{-1}(9.5144) = 84^\circ$$

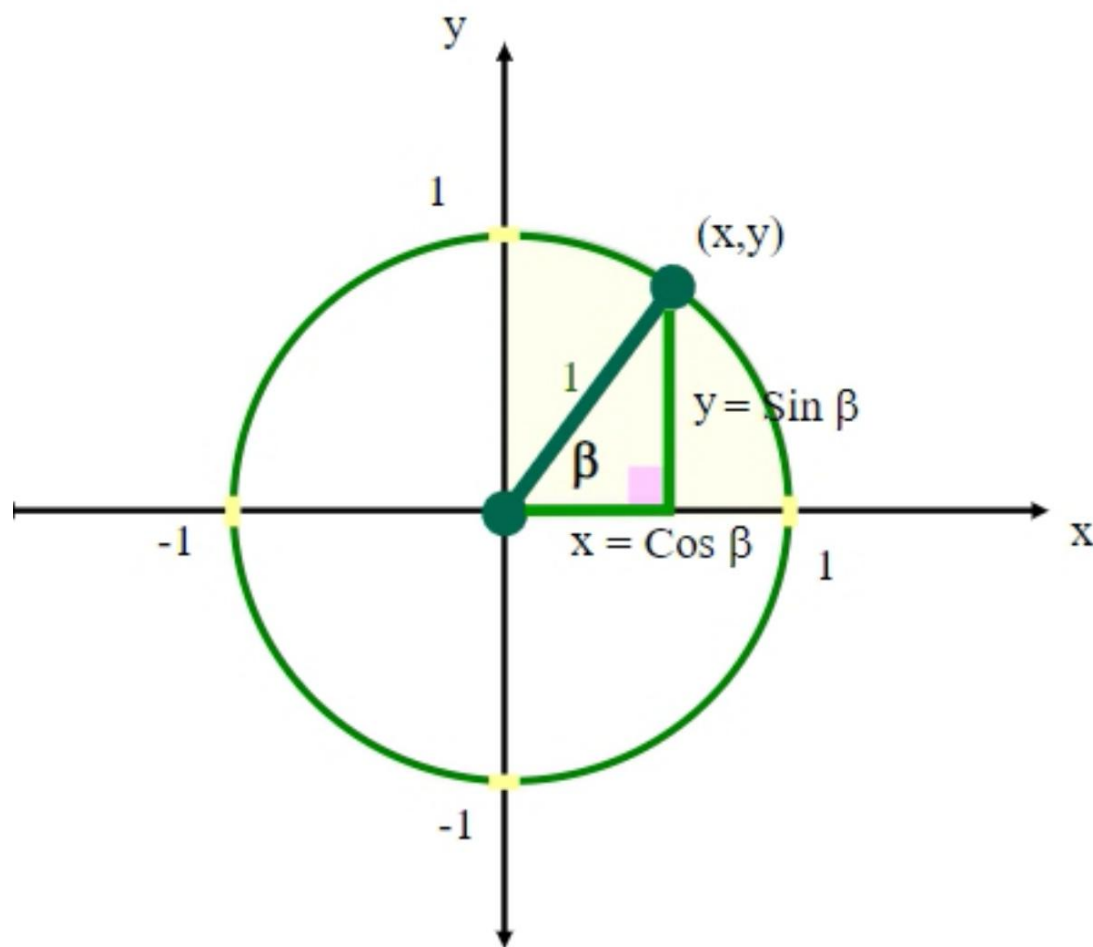
Ratio to Angle:

$$\text{acos}(0.9205) = 23^\circ$$

$$\text{asin}(0.0872) = 5^\circ$$

$$\text{atan}(9.5144) = 84^\circ$$

TRIGONOMETRIC CIRCLE: First Quadrant



$$\text{sine} = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\text{cosine} = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

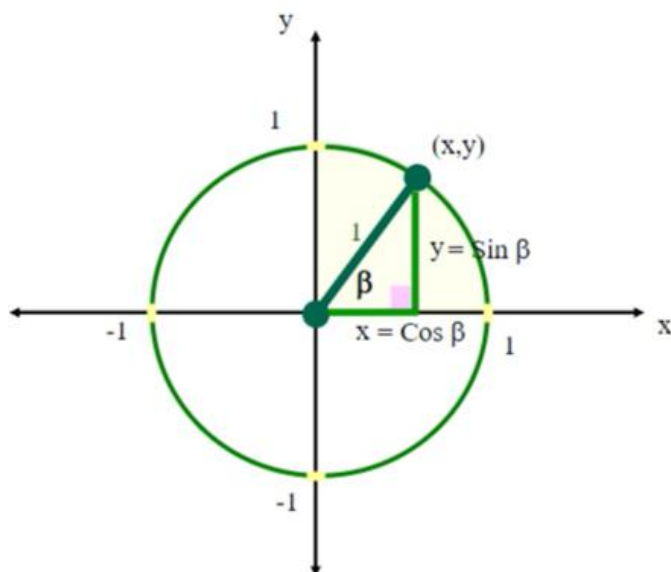
$$\text{tangent} = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\text{cotangent} = \frac{\text{adjacent side}}{\text{opposite side}}$$

$$\text{secant} = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\text{cosecant} = \frac{\text{hypotenuse}}{\text{opposite side}}$$

Trigonometric Circle



$$\sin \beta = \frac{y}{1}$$

$$\sin \beta = y$$

$$\cos \beta = \frac{x}{1}$$

$$\cos \beta = x$$

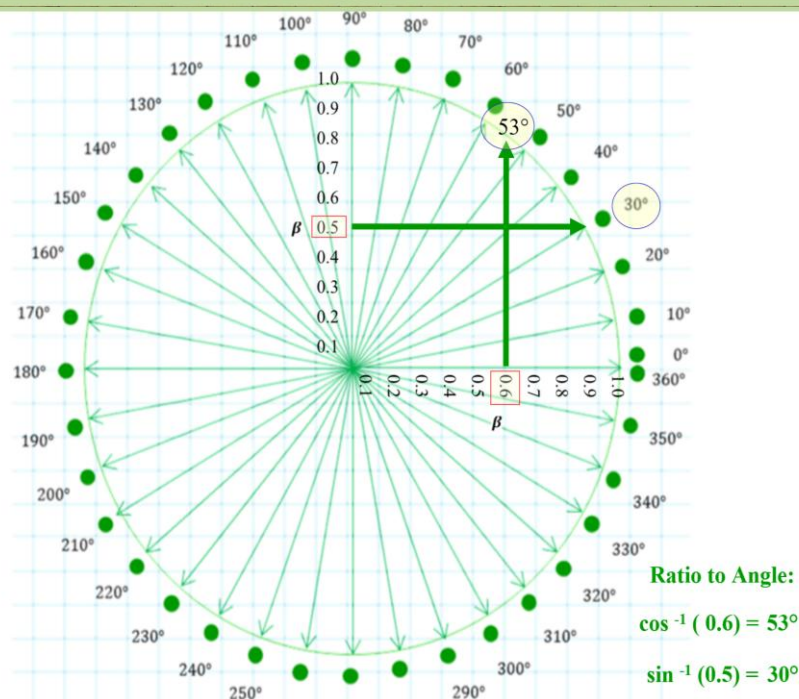
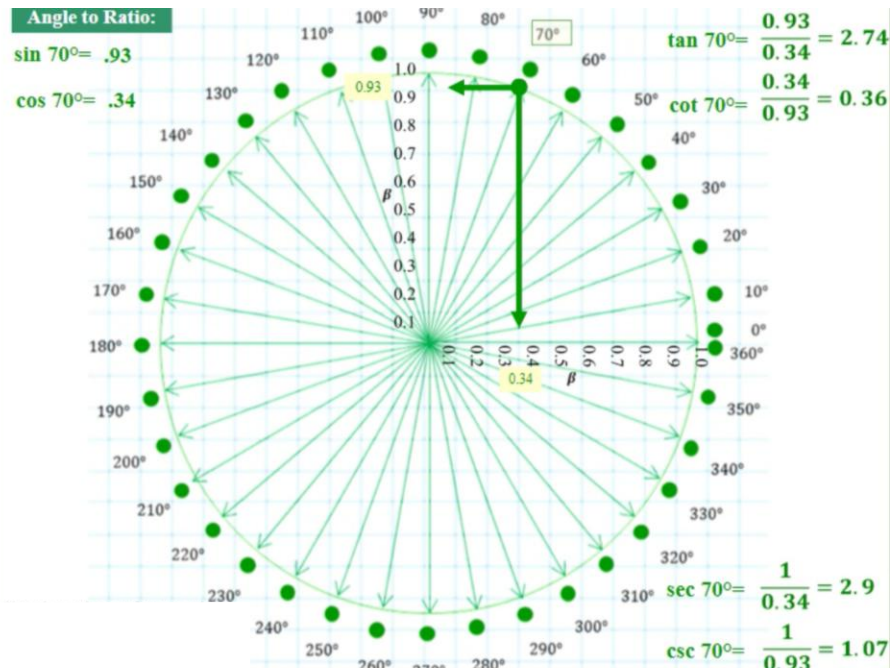
$$\tan \beta = \frac{y}{x}$$

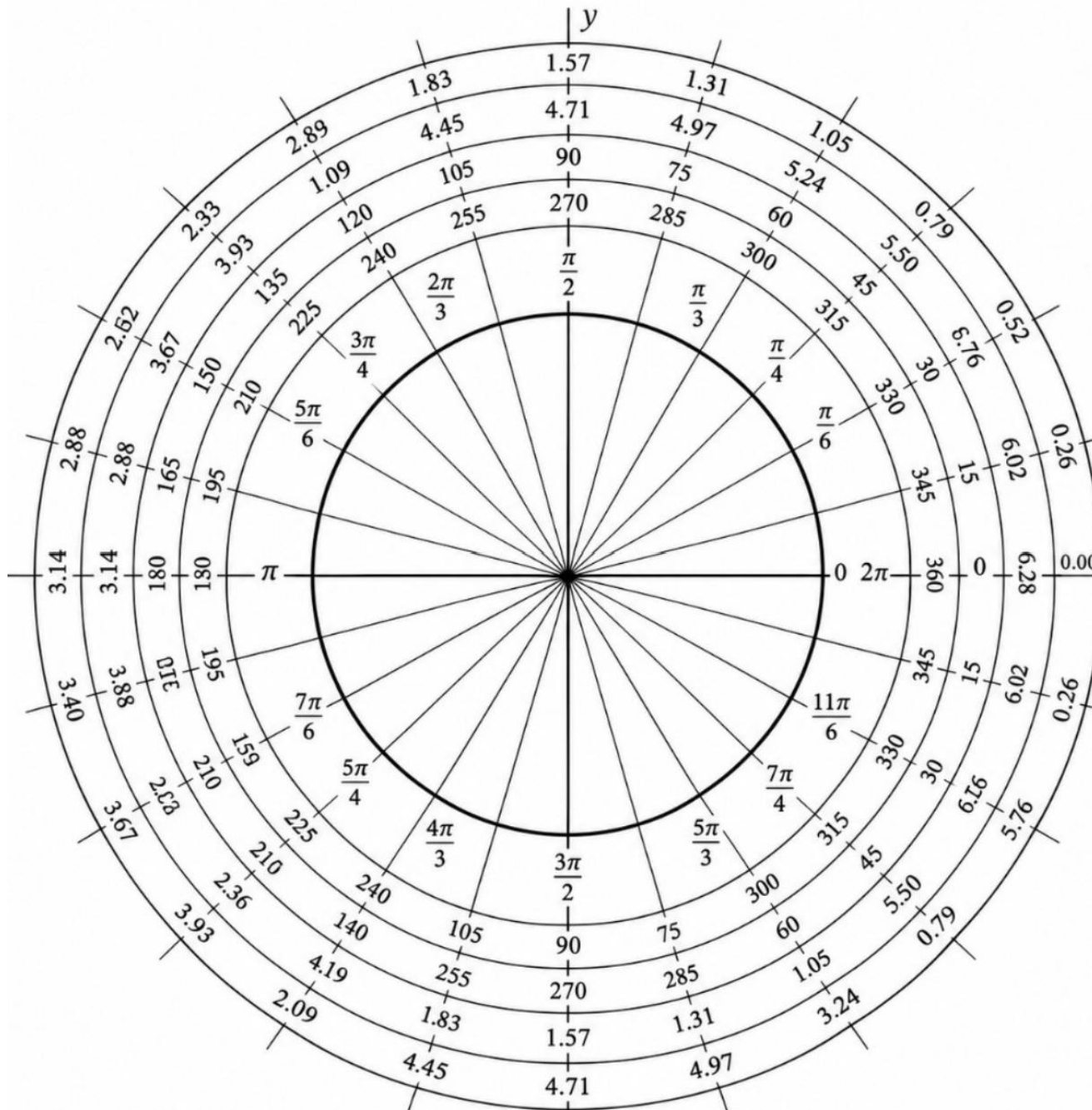
$$\cot \beta = \frac{x}{y}$$

$$\sec \beta = \frac{1}{x}$$

$$\csc \beta = \frac{1}{y}$$

Using the first quadrant to get all 6 trigonometric ratios





$$\frac{180^\circ}{180^\circ} = \frac{\pi}{180^\circ}$$

$$1^\circ = \frac{3.14}{180} \approx 0.018 \text{ Radians}$$

**Radians
and
Degrees**

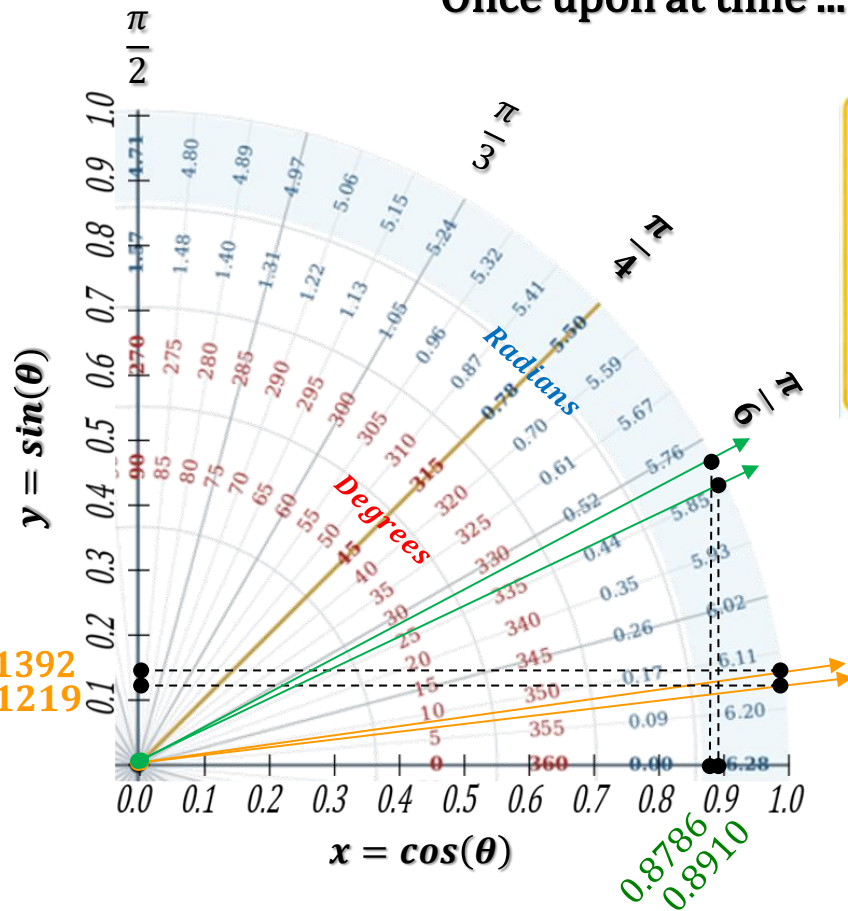
$$180^\circ = \pi$$

$$\frac{180^\circ}{\pi} = \frac{\pi}{\pi}$$

$$\frac{180^\circ}{3.14} = \frac{\pi}{\pi}$$

$$57.3^\circ = 1 \text{ Radian}$$

Once upon at time ... no calculators were available.



Example 1: Estimate $\sin(7.5^\circ)$

Use the table values $\sin(7^\circ) = 0.1219$ and $\sin(8^\circ) = 0.1392$.

$$\begin{array}{ccc} 0.1219 & x & 0.1392 \\ 7^\circ & 7.5^\circ & 8^\circ \end{array} \quad \frac{(7.5 - 7)}{(8 - 7)} = 0.5$$

$$\begin{aligned} \sin(7.5^\circ) &\approx 0.1219 + 0.5(0.1392 - 0.1219) \\ &\approx 0.1219 + 0.5(0.0173) \\ &\approx 0.13055 \approx \mathbf{0.1306} \end{aligned}$$

This is an estimate between 7° and 8° .

Example 2: Estimate $\cos(28^\circ)$

Use the table values $\cos(27^\circ) = 0.8910$ and $\cos(29^\circ) = 0.8746$.

$$\begin{array}{ccc} 0.8910 & x & 0.8746 \\ 27^\circ & 28^\circ & 29^\circ \end{array} \quad \frac{(28 - 27)}{(29 - 27)} = \frac{1}{2} = 0.5$$

$$\begin{aligned} \cos(28^\circ) &\approx 0.8910 + 0.5(0.8746 - 0.8910) \\ &\approx 0.8910 + 0.5(-0.0164) \\ &\approx \mathbf{0.8828} \end{aligned}$$

The exact value is close, so a small interpolation error is normal.

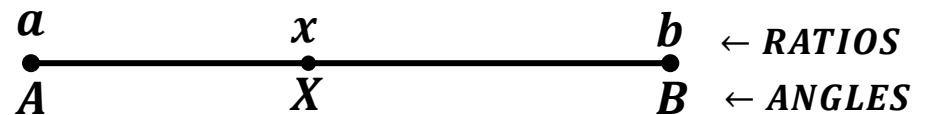
Table 1: 0° to 15° (by 1°)

ANGLE	SINE	COSINE
0	0.0000	1.0000
1	0.0175	0.9998
2	0.0349	0.9994
3	0.0523	0.9986
4	0.0698	0.9976
5	0.0872	0.9962
6	0.1045	0.9945
7	0.1219	0.9925
8	0.1392	0.9903
9	0.1564	0.9877
10	0.1736	0.9848
11	0.1908	0.9816
12	0.2079	0.9781
13	0.2250	0.9744
14	0.2419	0.9703
15	0.2588	0.9659

Table 2: 20° to 35° (by 1°)

ANGLE	SINE	COSINE
20	0.3420	0.9397
21	0.3584	0.9336
22	0.3746	0.9272
23	0.3907	0.9205
24	0.4067	0.9135
25	0.4226	0.9063
26	0.4384	0.8988
27	0.4540	0.8910
28	0.4695	0.8829
29	0.4848	0.8746
30	0.5000	0.8660
31	0.5150	0.8572
32	0.5299	0.8480
33	0.5446	0.8387
34	0.5592	0.8290
35	0.5736	0.8192

Many times a trigonometric value when retrieved from tables can not be taken directly from the table and makes necessary to use what we know from proportional reasoning to set up a proportion and perform an interpolation

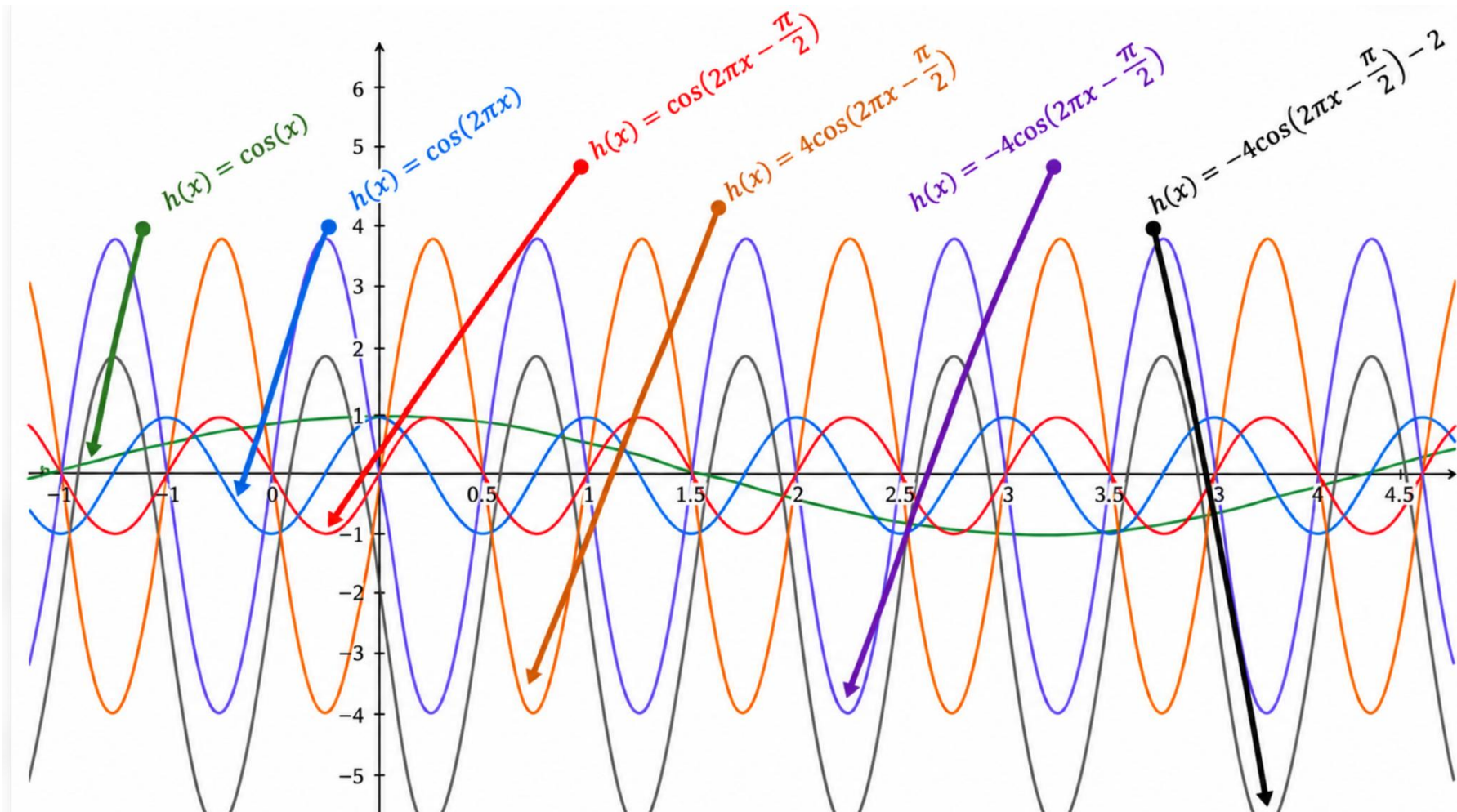


$$\frac{x - a}{X - A} = \frac{b - a}{B - A} \rightarrow x = a + (X - A) \left(\frac{b - a}{B - A} \right)$$

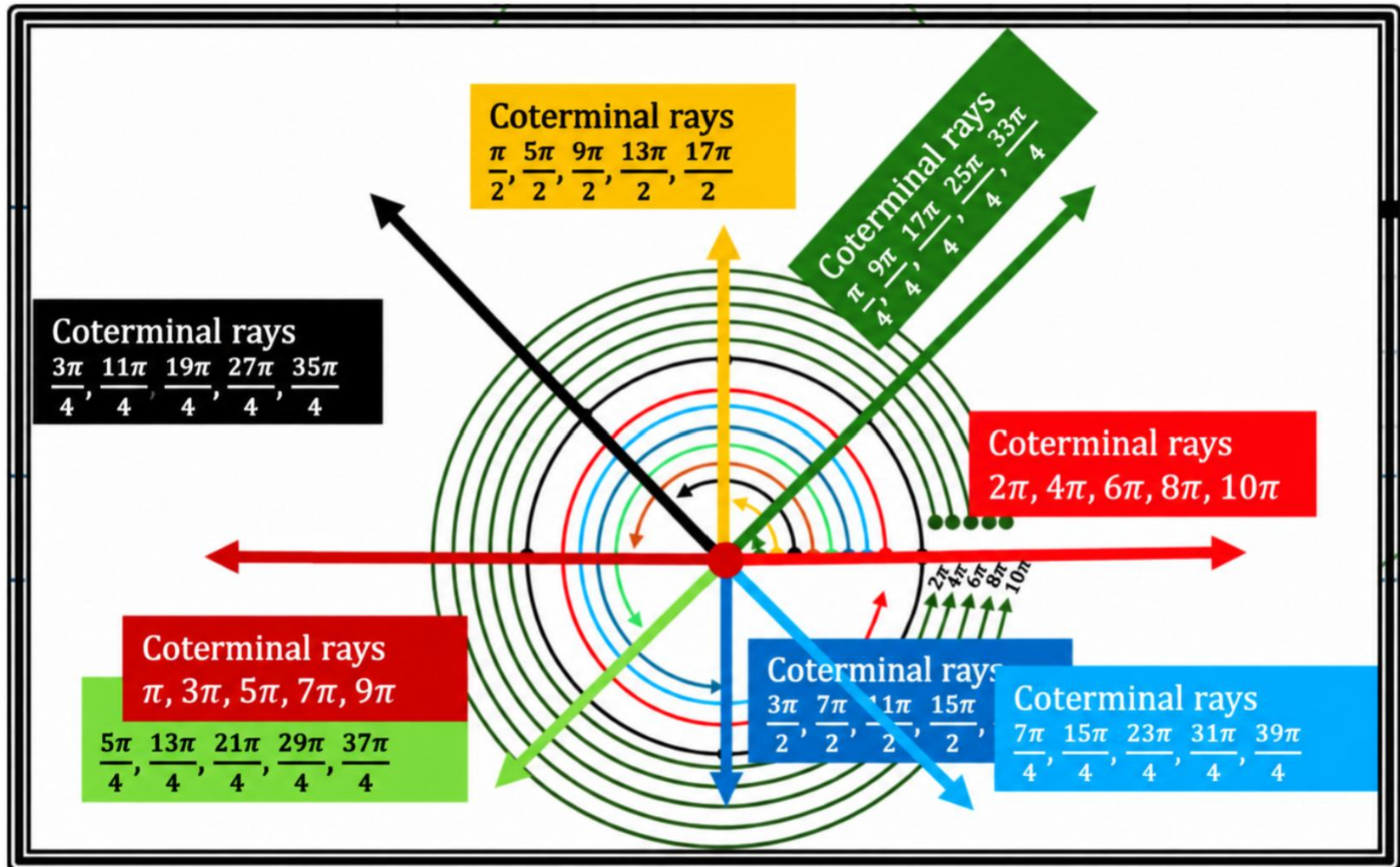
Common Trigonometric Values for Sine and Cosine

θ QUADRANT I				θ QUADRANT II				θ QUADRANT III				θ QUADRANT VI			
RADIANS	DEGREES	Sin	Cos	RADIANS	DEGREES	Sin	Cos	RADIANS	DEGREES	Sin	Cos	RADIANS	DEGREES	Sin	Cos
$\frac{\pi}{12}$	15°	.26	.97	$\frac{7}{12}\pi$	105°	.97	-.26	$\frac{13}{12}\pi$	195°	-.26	-.97	$\frac{19}{12}\pi$	285°	-.97	.26
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	.5	$\frac{2}{3}\pi$	120°	$\frac{\sqrt{3}}{2}$	-.5	$\frac{7}{6}\pi$	210°	-.5	$-\frac{\sqrt{3}}{2}$	$\frac{5}{3}\pi$	300°	$-\frac{\sqrt{3}}{2}$	.5
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	.71	$\frac{3}{4}\pi$	135°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{5}{4}\pi$	225°	$-\frac{\sqrt{2}}{2}$	-.71	$\frac{7}{4}\pi$	315°	$-\frac{\sqrt{2}}{2}$	.71
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	.87	$\frac{5}{6}\pi$	150°	.5	$-\frac{\sqrt{3}}{2}$	$\frac{4}{3}\pi$	240°	$-\frac{\sqrt{3}}{2}$	-.87	$\frac{11}{6}\pi$	330°	$-\frac{1}{2}$	.87
$\frac{5}{12}\pi$	75°	.97	.26	$\frac{11}{12}\pi$	165°	.26	-.97	$\frac{17}{12}\pi$	255°	-.97	-.26	$\frac{13}{12}\pi$	345°	-.26	.97
$\frac{\pi}{2}$	90°	1	0	π	180°	0	-1	$\frac{3}{2}\pi$	270°	-1	0	2π	360°	0	1

Effects of parameters in a trigonometric function:
 $f(x) = A \cdot \cos(B \cdot x + C) + D$



Terminal Rays and Periodic Concept



REFLECTED ON $x = \frac{\pi}{2}$

$$\sin(\pi - x) = \sin x$$

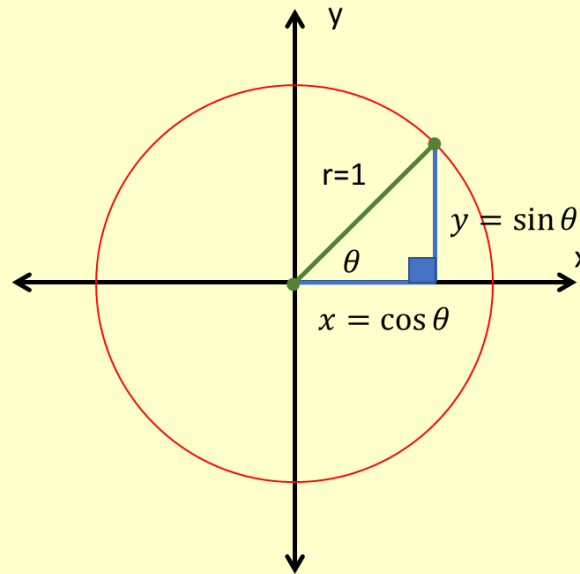
$$\cos(\pi - x) = -\cos x$$

$$\tan(\pi - x) = -\tan x$$

$$\cot(\pi - x) = -\cot x$$

$$\sec(\pi - x) = -\sec x$$

$$\csc(\pi - x) = \csc x$$



Pythagorean Theorem (Pythagorean Identities proof):

$$r^2 = x^2 + y^2 \Rightarrow 1 = \sin^2 x + \cos^2 x$$

$$\frac{1}{\sin^2 x} = \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}$$

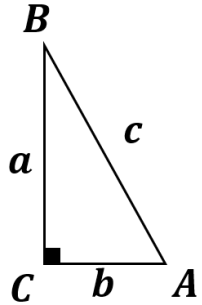
$$\csc^2 x = 1 + \cot^2 x$$

$$\frac{1}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}$$

$$\sec^2 x = \tan^2 x + 1$$

Distance Formula between two points in a plane:

Recall Pythagorean Theorem:



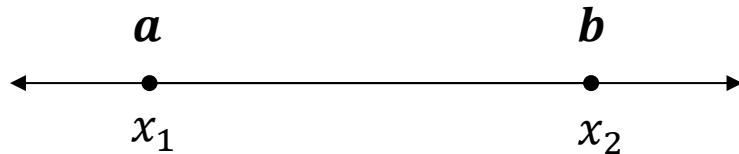
$$c^2 = a^2 + b^2$$

$$|c| = \sqrt{a^2 + b^2}$$

$$c = \pm \sqrt{a^2 + b^2}$$

(-) no sense here

and Recall:

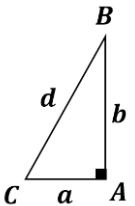
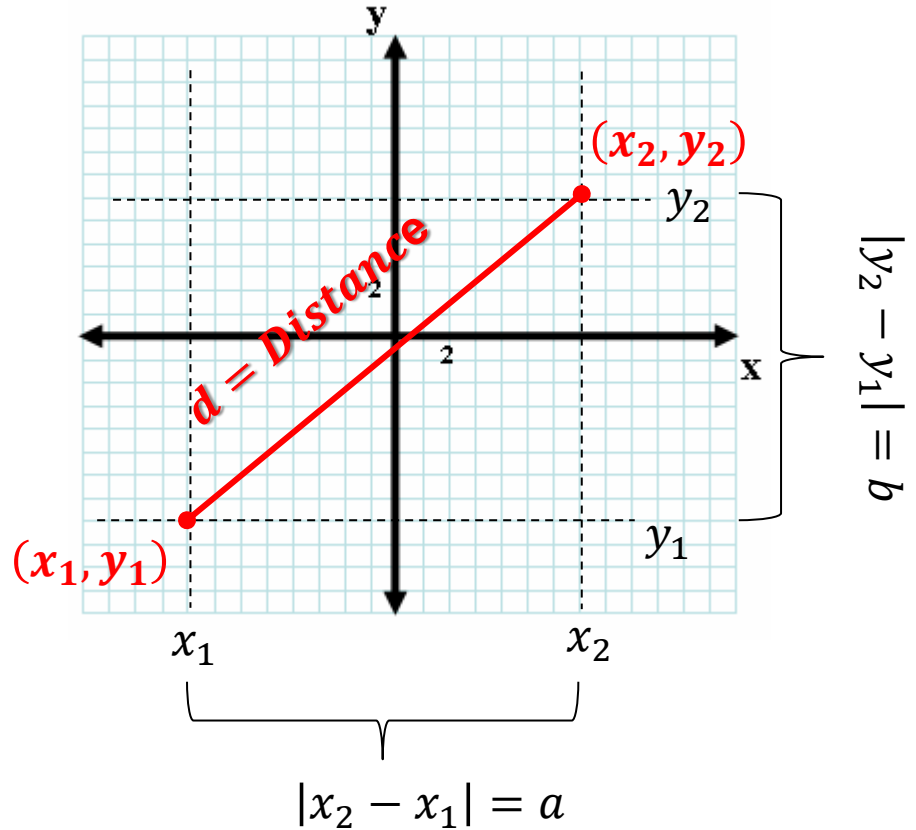


$$\text{Distance}_{a \text{ to } b} = |b - a|$$

$$|b - a| = \begin{cases} -(b - a), & (b - a) < 0 \\ (b - a), & (b - a) \geq 0 \end{cases}$$

and

$$\sqrt{x^2} = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$



Combining ideas at the left:

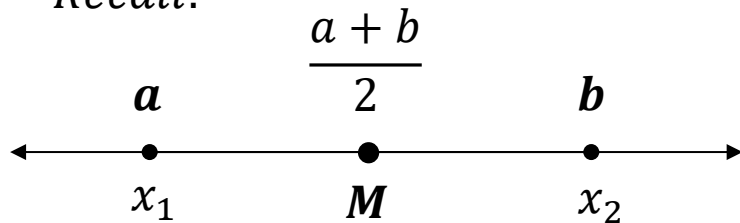
Distance between 2 points

(x_2, y_2) and (x_1, y_1) in the **coordinate plane**:

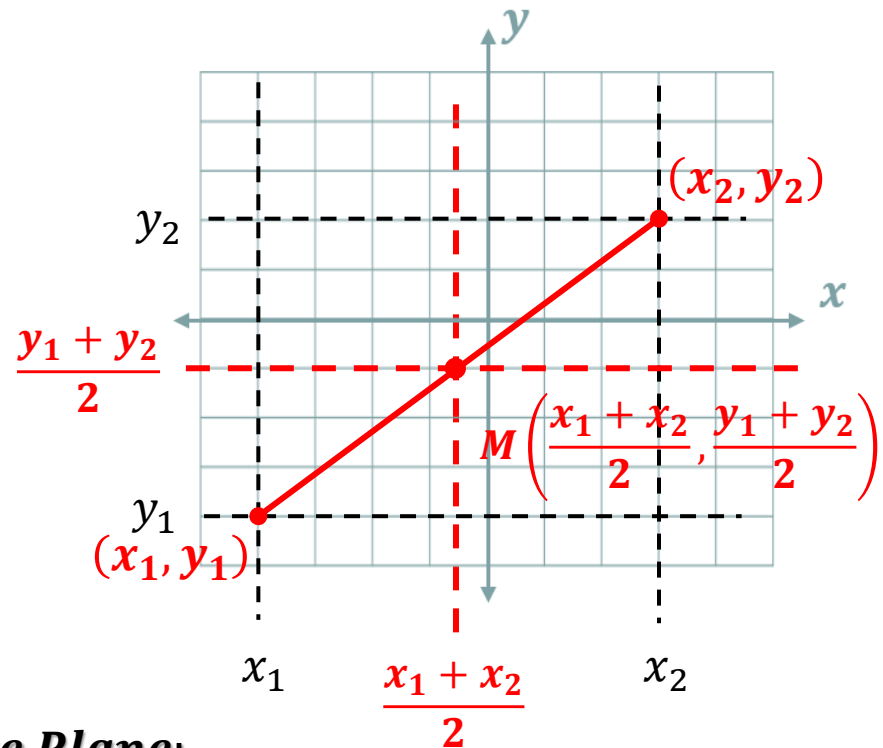
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint Formula between two points in a plane:

Recall:



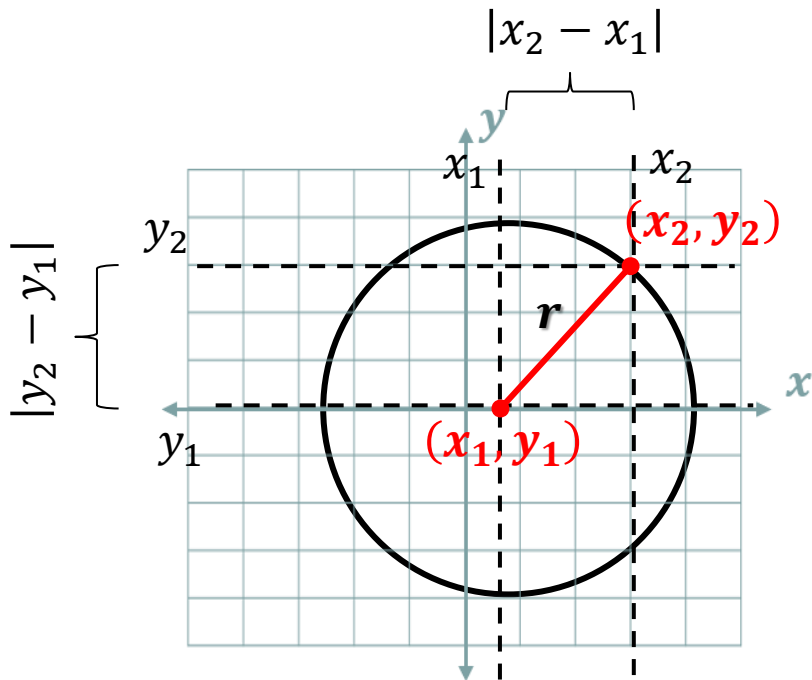
$$\text{Midpoint}_{a \text{ to } b} = \frac{a+b}{2}$$



Midpoint Formula in the Coordinate Plane:

$$M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Conics: Circle Formula



Applying Pythagorean Theorem

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = r^2$$

where:

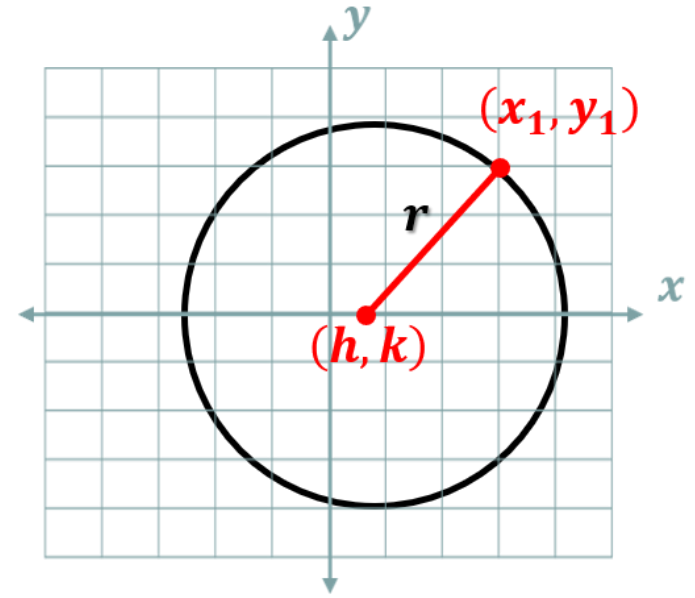
$(x_1, y_1) \rightarrow$ center of circle

$(x_2, y_2) \rightarrow$ point on the circumference

$r \rightarrow$ radius

or

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Circle Formula:

$$(x_1 - h)^2 + (y_1 - k)^2 = r^2$$

$(h, k) \rightarrow$ center of circle

$(x_1, y_1) \rightarrow$ point on circumference

$$r = \sqrt{(x_1 - h)^2 + (y_1 - k)^2}$$